A Strain-Structured SIS Model with Mutation: Spectral Thresholds and Long-Time Dynamics

Enriqué DEBRABANT[1]

[1]Institut Denis Poisson (IDP), Université de Tours, CNRS, France ,

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Abstract

We study the global dynamics of a strain-structured SIS model with a nonlocal mutation kernel. We characterize the epidemic threshold in terms of the spectral bound of the linearized operator and relate it to a generalized basic reproduction number $\mathcal{R}_0$, a formulation that remains valid even when the recovery rate vanishes locally. We show that the sign of the spectral bound determines extinction and persistence, and characterize the critical dynamics in the regular case. In the regular supercritical regime, we also construct an example with a unique unstable endemic equilibrium, showing that convergence towards the endemic state fails in general. Under an additional balanced transmission condition, however, we prove convergence towards the endemic equilibrium.

Keywords: SIS model, structured populations, spectral analysis, basic reproduction number, mutation-selection, stability.

1 Introduction

The mathematical modeling of infectious diseases has a long tradition, dating back to the pioneering work of Kermack and McKendrick [27] and later developed by Anderson and May [3]. Among the classical compartmental models, the SIS (Susceptible–Infected–Susceptible) model is of special interest for infections which do not confer long-term immunity after recovery. In its simplest form, the SIS model reads

$$\begin{cases} \frac{d}{dt}S(t) = \gamma I(t) - \beta S(t)I(t), \\ \frac{d}{dt}I(t) = \beta S(t)I(t) - \gamma I(t), \end{cases}$$

where $S(t)$ and $I(t)$ denote the proportions of susceptible and infected individuals, respectively. This model assumes that all infected individuals are epidemiologically identical.

In most biological situations, this assumption seems to be rather questionable. Indeed, infected individuals may differ significantly in terms of susceptibility, infectivity, and recovery rates due to genetic or phenotypic variability, or the coexistence of multiple pathogen strains. Such heterogeneity can strongly influence the epidemic dynamics and lead to behaviors that cannot be captured by homogeneous models. This observation has motivated the development of structured SIS models, where the infected population is described by a distribution over a continuous strain variable.

Early heterogeneous SIS models with multiple interacting groups were introduced by Lajmanovich and Yorke [30], and can be viewed as finite-dimensional precursors of the continuous structured models considered by Delmas, Dronnier and Zitt [17].

To account for the heterogeneity, we consider a structured SIS model where the infected population is described by a density $I(t, x)$ depending on a continuous strain variable $x \in \Omega$ that

characterizes the pathogen. Unlike in [17, 30], where the trait structure is a characteristic of the host, in our study the susceptible population depends only on t.

The dynamics is given by the system

$$\begin{cases} \frac{d}{dt}S(t) = \int_\Omega \gamma(y)I(t,y)\,dy - S(t)\int_\Omega\int_\Omega k(x,y)I(t,y)\,dx\,dy, \\ \partial_t I(t,x) = S(t)\int_\Omega k(x,y)I(t,y)\,dy - \gamma(x)I(t,x), \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is a compact set of positive Lebesgue measure representing the strain space, $k(x,y)$ models the transmission and mutation process, and $\gamma(x)$ denotes the recovery rate.

Summing the two equations, we obtain that the total population is conserved. Without loss of generality, we can normalize to one, that is $S(t) + \int_\Omega I(t,x)dx = 1$ for all time t.

Substituting this expression into the equation for I, we obtain the following closed integro-differential equation:

$$\partial_t I(t,x) = \left(1 - \int_\Omega I(t,y)\,dy\right)\int_\Omega k(x,y)I(t,y)\,dy - \gamma(x)I(t,x). \tag{2}$$

We make the following structural assumptions on the model:

(K1) Regularity of the transmission kernel. The kernel $k(x,y)$ is non-negative and the mapping $y \mapsto k(\cdot,y)$ belongs to $\mathcal{C}(\Omega; L^1(\Omega))$.
This assumption guarantees an appropriate compactness property for the associated integral operator. Similarly, it permits to define :

$$\overline{k} := \sup_{y\in\Omega}\int_\Omega k(x,y)\,dx < +\infty.$$

(K2) Irreducibility on $L^1(\Omega)$. For every measurable set $S \subset \Omega$ satisfying

$$0 < \mu(S) < \mu(\Omega),$$

where μ denotes the Lebesgue measure, one has

$$\int_{\Omega\setminus S}\int_S k(x,y)\,d\mu(y)\,d\mu(x) > 0.$$

(K3) Strong positivity (Irreducibility on $\mathcal{C}(\Omega)$.) For any nonempty open subset $A \subset \Omega$,

$$\int_A k(x,y)\,dx > 0, \qquad \forall y \in \Omega.$$

These conditions ensure that the strain space cannot split into dynamically isolated components.

(Γ1) Recovery rate. The function γ is continuous and non-negative on Ω. We set

$$\underline{\gamma} := \min_{x\in\Omega}\gamma(x) \qquad \text{and} \qquad \overline{\gamma} := \|\gamma\|_\infty$$

.

A key focus of this work is the singular case $\underline{\gamma} = 0$, which allows for the existence of strains associated with absent recovery.

When the recovery rate is uniformly positive, our model is a particular case of the measure-valued selection–mutation framework studied by Ackleh, Cleveland and Thieme [1]. Indeed, defining $\beta(y) = \int_\Omega k(x,y)\,dx$ and the probability kernel $P(y,dx) = k(x,y)\,dx/\beta(y)$, our equation corresponds to birth and mortality rates $B(m,y) = (1-m)_+\beta(y)$ and $D(m,y) = \gamma(y)$, respectively. The positive-part extension agrees with the SIS dynamics on the invariant region $0 \le m \le 1$. Under our kernel assumptions, their sufficient condition for persistence implies $s(L_0) > 0$, but the converse fails. Our spectral criterion therefore establishes persistence for additional models within this class. We also investigate the singular regime in which the recovery rate may vanish, which lies outside their uniformly positive mortality assumption.

Proposition 1.1 (Well-posedness and invariance)**.** *Under assumptions (K1) and (Γ1), for any initial condition $I_0 \in L^1_+(\Omega)$ such that*

$$m(0) := \int_\Omega I_0(x)\,dx \in [0,1),$$

there exists a unique global solution

$$I \in C^1(\mathbb{R}_+; L^1(\Omega))$$

to the system. Moreover, the solution remains nonnegative for all times and the total mass

$$m(t) := \int_\Omega I(t,x)\,dx \qquad \textit{satisfies} \qquad m(t) \in [0,1], \quad \forall t \ge 0.$$

To capture the nonlinear feedback induced by the depletion of susceptibles, we introduce, for each $m \in [0,1]$, the linear operator $L_m : L^1(\Omega) \to L^1(\Omega)$ defined by

$$L_m\phi(x) = (1-m)\int_\Omega k(x,y)\phi(y)\,dy - \gamma(x)\phi(x).$$

This operator describes the linear infection dynamics when the total infected mass is frozen at the level m.

The system can admit, besides the disease-free equilibrium ($I \equiv 0$), nontrivial stationary states corresponding to the persistence of the infection. In the regular case, such stationary states are described by L^1 densities. However, when the recovery rate vanishes on part of the trait space, stationary states may develop concentration on the zero-recovery set. For this reason, we use a measure formulation of endemic equilibria. The possible nonexistence of a principal eigenfunction and the emergence of singular eigenmeasures for nonlocal operators have been investigated by Bürger and Bomze [7], by Coville [5, 15] and Cloez and Gabriel [14]. Related concentration phenomena have also been studied in an epidemiological model with continuous phenotypes [24].

More precisely, denoting by $M_+(\Omega)$ the space of the positive measure, if $\mu \in \mathcal{M}_+(\Omega)$, we define

$$K\mu := \rho_\mu(x)\,dx, \qquad \rho_\mu(x) := \int_\Omega k(x,y)\,\mu(dy), \qquad \rho_\mu \in L^1(\Omega),$$

and $\Gamma\mu := \gamma\,\mu$. Thus the same expression defining L_m also has a natural meaning on positive Radon measures:

$$L_m\mu := (1-m)K\mu - \Gamma\mu.$$

This should be understood only as the measure extension of the formula above.

An *endemic measure* is a measure

$$\mu^* \in \mathcal{M}_+(\Omega) \quad \text{with mass} \quad m^* := \mu^*(\Omega) > 0,$$

satisfying

$$0 = (1 - m^*) \left(\int_\Omega k(x, y)\, \mu^*(dy) \right) dx - \gamma(x)\, \mu^*(dx).$$

With the above notation, this can be written compactly as $L_{m^*}\mu^* = 0$ in the sense of measures.

In the regular case ($\underline{\gamma} > 0$), this notion reduces to the usual L^1 notion of endemic equilibrium. Indeed, since γ is bounded away from zero, the identity

$$\gamma\, \mu^* = (1 - m^*) K \mu^*$$

implies that μ^* is absolutely continuous with respect to the Lebesgue measure. Hence $\mu^*(dx) = I^*(x)\, dx$ for some $I^* \in L^1_+(\Omega)$, and the stationary equation becomes

$$0 = (1 - m^*) \int_\Omega k(x, y) I^*(y)\, dy - \gamma(x) I^*(x), \qquad m^* = \int_\Omega I^*(x)\, dx.$$

Thus endemic equilibria may be described by densities in the regular regime, whereas the measure formulation is needed in the singular regime in order to capture possible concentration phenomena.

The family $(L_m)_{m\in[0,1]}$ therefore provides the natural spectral framework for studying stationary states. For a fixed value of m, the stationary problem is linear, and the admissible masses are selected by the values of m for which the eigenvalue problem

$$L_m \mu = 0$$

admits a nontrivial positive solution in the measure sense.

This motivates the study of the equation $s(L_m) = 0$, where $s(L_m)$ denotes the spectral bound of L_m as an operator on $L^1(\Omega)$:

$$s(L_m) := \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(L_m)\}.$$

Here $\sigma(L_m)$ denotes the spectrum of L_m on $L^1(\Omega)$. Throughout the paper, unless explicitly stated otherwise, $s(L_m)$ always refers to this L^1 spectral bound. The use of Radon measures concerns the eigenvectors associated with the measure extension of L_m, not the introduction of a second spectral bound on $\mathcal{M}(\Omega)$.

As will be shown below, the map $m \mapsto s(L_m)$ is continuous and non-increasing. Its zeros therefore provide the compatibility condition selecting the possible masses of endemic states. In this sense, the condition

$$s(L_m) = 0$$

plays the role of a nonlinear spectral constraint for the existence of endemic equilibria.

Specifically, the linearization of the system at the disease-free equilibrium corresponds to the operator L_0, which describes the initial growth or decay of small perturbations of the disease-free-equilibrium.

The global behavior of the nonlinear system is therefore expected to be governed by the spectral properties of the linearized operator L_0. In particular, the sign of the spectral bound $s(L_0)$ is expected to determine whether small perturbations around the disease-free equilibrium are growing or decaying, and thus serves as a threshold parameter for the epidemic dynamics.

In classical epidemic models, the threshold phenomena are usually formulated in terms of the basic reproduction number $\mathcal{R}_0$, which represents the average number of secondary infections caused

by a typical infected individual in a completely susceptible population. In structured population models, this quantity is typically defined through the next-generation operator [19, 39]

$$\mathcal{N} = K\Gamma^{-1}.$$

where the transmission operator $K : L^1(\Omega) \to L^1(\Omega)$ and the loss operator $\Gamma : L^1(\Omega) \to L^1(\Omega)$ are given by

$$K\phi(x) := \int_\Omega k(x,y)\phi(y)\,dy, \qquad \Gamma\phi(x) := \gamma(x)\phi(x).$$

However, this classical framework is based on the invertibility of the loss operator Γ. When the recovery rate is allowed to vanish locally, the operator Γ^{-1} is no longer well defined on $L^1(\Omega)$, and the usual definition of $\mathcal{R}_0$ is not applicable. To address this issue, we use the following generalized definition:

$$\mathcal{R}_0 := \lim_{\lambda \downarrow 0} r\Big(K(\lambda I + \Gamma)^{-1}\Big),$$

where $r(\cdot)$ denotes the spectral radius, which is defined for a bounded operator A, by:

$$r(A) := \sup\left\{|\mu| \;|\mu \in \sigma(A)\right\}.$$

This definition coincides with $r(\mathcal{N})$ when $\underline{\gamma} > 0$, and remains meaningful in the singular case $\underline{\gamma} = 0$, where it defines a quantity in $[0, +\infty]$. In such regimes, the spectral bound $s(L_0)$ (where $L_0 = K - \Gamma$) remains well defined and continues to govern the growth or decay of small perturbations around the disease-free equilibrium.

This spectral characterization provides the relevant invasion criterion. It then allows us to distinguish extinction and persistence regimes, with a critical regime whose behavior depends on whether the recovery rate is uniformly positive or may vanish.

Theorem 1.2 (Threshold characterization)**.** • ***Regular case*** **(*$\underline{\gamma} > 0$*).** *The spectral bound $s(L_0)$ and the basic reproduction number $\mathcal{R}_0 = r\left(K\Gamma^{-1}\right)$ provide equivalent threshold criteria, in the sense that*

$$s(L_0) > 0 \iff \mathcal{R}_0 > 1, \qquad s(L_0) < 0 \iff \mathcal{R}_0 < 1.$$

• ***Singular case*** **(*$\underline{\gamma} = 0$*).** *Even when the recovery rate vanishes locally and the classical next-generation operator may fail to be well defined, the threshold relation between $s(L_0)$ and $\mathcal{R}_0$ remains valid in the sense that*

$$s(L_0) > 0 \iff \mathcal{R}_0 > 1, \qquad s(L_0) = 0 \iff \mathcal{R}_0 \leq 1.$$

The first statement is a consequence of the general spectral theory developed by Thieme [37], whereas the second one is established in the present work.

This spectral characterization offers the correct linear threshold criterion for the model. The primary aim is then to employ this information to study the complete nonlinear dynamics of the SIS model. Specifically, we show that the sign of $s(L_0)$ determines the long-time dynamics of solutions, yielding a clean dichotomy between extinction and persistence of the infection.

Theorem 1.3 (Global dynamics and equilibria)**.** *Recall that we denote by $m(t) = \int_\Omega I(t,y)dy$ the total mass of infected individuals, which satisfies $0 \leq m(t) \leq 1$.*

- ***Extinction*** **(*$s(L_0) < 0$*).** *The disease-free equilibrium is globally asymptotically stable and is the unique equilibrium of the system. Moreover, there exist constants $C, \omega > 0$ such that*

  $$m(t) \leq Ce^{-\omega t}m(0) \qquad \text{for all } t \geq 0,$$

  so that $m(t) \to 0$ exponentially fast as $t \to +\infty$.

- ***Critical case*** *(*$s(L_0) = 0$*).* *If the recovery rate is uniformly positive (i.e.* $\underline{\gamma} > 0$*), the disease-free equilibrium is still globally attracting, with algebraic decay:*

$$m(t) = \mathcal{O}(t^{-1}) \qquad \text{as } t \to +\infty.$$

- ***Persistence*** *(*$s(L_0) > 0$*).* *The disease-free equilibrium is unstable and the infection persists in the population:*

$$\liminf_{t\to+\infty} m(t) > 0.$$

In this regime, the system admits endemic equilibria. If $\underline{\gamma} > 0$*, this equilibrium is unique. In contrast, in the singular case* $\underline{\gamma} = 0$*, the set of equilibrium states may fail to be unique.*

Remark 1.4 (Threshold $m_\star$ in the singular case). In the singular case $\underline{\gamma} = 0$, define $m_\star := \inf\{m \in [0,1] : s(L_m) = 0\}$. Assume that there exists a compact set $A \subset \Omega$ with positive measure such that

$$a_A := \inf_{y\in A} \int_A \frac{k(x,y)}{\gamma(x)}\, dx > 1.$$

This sufficient criterion already appears in [31, Theorem 2.1], where it ensures the existence of a principal eigenvalue for nonlocal operators acting on $C(\overline{\Omega})$. In our L^1 framework, Proposition 2.3 gives $s(L_m) > 0$ for all $m < \frac{a_A - 1}{a_A}$, and therefore $m_\star \geq \frac{a_A-1}{a_A} > 0$.

Conversely, if

$$b := \sup_{y\in\Omega} \int_\Omega \frac{k(x,y)}{\gamma(x)}\, dx \in (1, +\infty),$$

then $s(L_m) = 0$ for all $m > \frac{b-1}{b}$, so that $m_\star \leq \frac{b-1}{b} < 1$. This upper bound follows from the norm criterion discussed in [7, p. 244], combined with their Corollary 3.8.

Hence, if both conditions hold, $0 < m_\star < 1$.

In particular, the map $m \mapsto s(L_m)$ reaches the value 0 strictly before $m = 1$ and remains constant thereafter, producing a non-trivial spectral plateau and, whenever the corresponding positive eigenelements exist, a continuum of endemic equilibria.

We now turn to a finer description of endemic equilibria in the singular regime where the recovery rate may vanish. While the previous result characterizes the persistence of the infection, it does not provide information on the structure of the equilibrium measure. In particular, the possible emergence of singular components requires a more detailed analysis. The following result shows that any such singularities are necessarily concentrated on the set of strains with vanishing recovery rate, and provides a sharp criterion for the presence or absence of mass on this set.

Definition 1.5 (Vanishing recovery set and infectious potential)**.** We define the vanishing recovery set by

$$Z := \{x \in \Omega : \gamma(x) = 0\}.$$

For $y \in \Omega$, we define the infectious potential associated with the non-vanishing recovery region by

$$\mathcal{B}(y) := \int_{\Omega\setminus Z} \frac{k(x,y)}{\gamma(x)}\, dx \in [0, +\infty],$$

and we set

$$\overline{\mathcal{B}} := \sup_{y\in\Omega} \mathcal{B}(y).$$

Theorem 1.6 (Concentration on the vanishing recovery set)**.** *Let μ be an equilibrium measure with total mass $m \in (0,1)$, and*

$$\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sing}}$$

be the Lebesgue decomposition of μ with respect to the Lebesgue measure, where $\mu_{\mathrm{ac}} \ll dx$ and $\mu_{\mathrm{sing}} \perp dx$. Then the singular part of μ is necessarily supported on Z, that is

$$\mathrm{supp}(\mu_{\mathrm{sing}}) \subset Z.$$

Moreover, the following dichotomy holds:

- *If $\mathcal{B}(y) = +\infty$ for all $y \in Z$, then $\mu(Z) = 0$.*
- *If $0 < \overline{\mathcal{B}} < +\infty$ and $m > \frac{\overline{\mathcal{B}}-1}{\overline{\mathcal{B}}}$, then $\mu(Z) > 0$. In particular, if $|Z| = 0$, then $\mu_{\mathrm{sing}} \neq 0$.*

To further investigate the long-time dynamics in the endemic regime, we introduce an additional structural assumption on the transmission kernel, namely

$$\int_\Omega \frac{k(x,y)}{\gamma(x)}\,dx = \mathcal{R}_0, \qquad y \in \Omega.$$

This condition can be interpreted as a balance property of the infectious potential carried by each strain. Although restrictive, it still covers a broad class of transmission kernels and is substantially more general than the particular case $k(x,y) = Q(x)$.

Under this assumption, we construct a Lyapunov functional and prove that, in the supercritical regime $\mathcal{R}_0 > 1$, the total infected mass converges to

$$m^* = 1 - \frac{1}{\mathcal{R}_0},$$

and the full infected profile converges in $L^1(\Omega)$ towards the unique endemic equilibrium.

This limitation is genuine: we construct in Section 3 a regular supercritical example whose unique endemic equilibrium is unstable, thereby ruling out global convergence under the general assumptions of the model.

Relation to existing literature

A natural connection with the literature on selection–mutation models appears by first considering the mutation-free version of our equation. Formally, if an individual infected with strain x can only transmit the same strain, then

$$\partial_t I(t,x) = (\beta(x)S(t) - \gamma(x))I(t,x), \qquad S(t) = 1 - \int_\Omega I(t,y)\,dy.$$

Equivalently,

$$\partial_t I(t,x) = \left(a(x) - \int_\Omega b(x,y)I(t,y)\,dy\right) I(t,x),$$

with

$$a(x) = \beta(x) - \gamma(x), \qquad b(x,y) = \beta(x).$$

Hence, in the absence of mutation, our equation belongs to the class of competitive selection models discussed in [33, Chapter 2] and studied, for instance, in [18, 26]. Related long-time selection and concentration phenomena in the presence of multiple fitness peaks are investigated in [32]. In

this formulation, the different pathogen strains compete indirectly through their common depletion of the susceptible population. The same mutation-free equation is closely related to structured epidemic models in which a continuum of pathogen variants competes for susceptible hosts [20, 10]. In particular, Burie, Ducrot and Griette [8, 9] consider equations in which the infected population satisfies

$$\partial_t I(t,x) = (\beta(x)S(t) - \gamma(x))I(t,x),$$

while the susceptible population evolves according to a separate demographic equation. The selection mechanism acting on the pathogen trait is therefore very similar to the one arising in our mutation-free model. The dynamics of the susceptible population is, however, different. In our SIS setting, recovered individuals immediately return to the susceptible compartment and the conservation of the total population gives

$$S(t) = 1 - \int_\Omega I(t,x)\,dx.$$

By contrast, models including recruitment, natural mortality or a removed compartment lead to an independent evolution equation for S. Thus, despite their closely related selection mechanisms, neither framework is in general a particular case of the other.

Mutation mechanisms based on nonlocal kernels are classical in population genetics and mutation–selection theory. Continuum-of-alleles models [28] and their extensions describe transitions from a parental trait to an offspring trait by means of integral kernels [29, 14, 13, 23]; see, among others, Burger [6], Burger and Bomze [7], Calsina and Cuadrado [12, 11], and Raoul [34]. In a common continuous-time formulation, selection and mutation are represented as distinct mechanisms, leading schematically to equations of the form [5, 2]

$$\partial_t u = \left(a(x) - \int_\Omega b(x,y)u(t,y)\,dy\right)u + \mathcal{M}u,$$

where $\mathcal{M}$ is a linear mutation operator, for instance a nonlocal integral operator or a diffusion operator in trait space [4].

The present model uses the same classical idea of describing trait changes by a nonlocal kernel, but incorporates it into the epidemic transmission mechanism:

$$\partial_t I(t,x) = S(t)\int_\Omega k(x,y)I(t,y)\,dy - \gamma(x)I(t,x).$$

Here $k(x,y)$ represents the production of new infections carrying strain x from individuals infected with strain y. For instance, whenever the kernel is written in the form

$$k(x,y) = \beta(y)M(x,y), \qquad \int_\Omega M(x,y)\,dx = 1,$$

the quantity $\beta(y)$ represents the transmission rate of strain y, whereas $M(\cdot,y)$ describes the distribution of the strain carried by a newly infected individual. The mutation-free case is formally recovered for

$$M(dx,y) = \delta_y(dx).$$

Organization of the paper. The paper is organized as follows. In Section 2, we develop the spectral analysis of the family of operators $(L_m)_{m\in[0,1]}$ and establish the main properties of the spectral bound. Section 3 is devoted to the study of equilibrium states and the proof of the main results on the global dynamics. Finally, numerical simulations illustrating the theoretical results are presented at the end of this paper.

2 Spectral analysis and stationary states

In this section, we study the spectral properties of the family of operators

$$L_m = (1-m)K - \Gamma, \qquad m \in (-\infty, 1],$$

and use them to characterize endemic equilibria.

We first analyze the operator L_m for a fixed value of m, establishing compactness, positivity, and criteria ensuring either a spectral gap or saturation at the edge of the spectrum. We then study the spectral curve

$$m \longmapsto s(L_m),$$

including its monotonicity, continuity, and possible plateau behavior. Finally, we relate the zeros of this spectral curve to endemic equilibria and introduce a measure framework in order to describe possible singular stationary states when the recovery rate vanishes.

2.1 Fixed mass spectral analysis

Lemma 2.1. *The operator $K : L^1(\Omega) \to L^1(\Omega)$ is compact. Furthermore, the adjoint operator $K^* : L^\infty(\Omega) \to L^\infty(\Omega)$, given by $K^*f(y) = \int_\Omega k(x,y)f(x)\,dx$, maps $L^\infty(\Omega)$ into $C(\Omega)$.*

Proof. We first prove that K is compact on $L^1(\Omega)$. By assumption (K1), the map $y \longmapsto k(\cdot, y)$ is continuous from Ω into $L^1(\Omega)$. Since Ω is compact, this continuity is uniform. Hence, using the strong continuity of translations in $L^1(\mathbb{R}^d)$, we obtain $\sup_{y\in\Omega} \|\tau_h k(\cdot,y) - k(\cdot,y)\|_{L^1} \xrightarrow[h\to 0]{} 0$. Let B denote the unit ball of $L^1(\Omega)$. For every $\phi \in B$, one has $\|\tau_h(K\phi) - K\phi\|_{L^1} \le \sup_{y\in\Omega} \|\tau_h k(\cdot,y) - k(\cdot,y)\|_{L^1} \|\phi\|_{L^1}$. Therefore, $\sup_{\phi\in B} \|\tau_h(K\phi) - K\phi\|_{L^1} \xrightarrow[h\to 0]{} 0$. Moreover, $K(B)$ is bounded in $L^1(\Omega)$ and its elements are supported in the compact set Ω. By the Riesz–Fréchet–Kolmogorov compactness theorem, $K(B)$ is relatively compact in $L^1(\Omega)$. Hence K is compact. We now prove the regularity property of the adjoint operator. Let $f \in L^\infty(\Omega), y \in \Omega$ and $y_n \to y$ in Ω. Then, by (K1), $\|k(\cdot,y_n) - k(\cdot,y)\|_{L^1} \to 0$. Consequently, $|K^*f(y_n) - K^*f(y)| \le \|f\|_{L^\infty} \|k(\cdot,y_n) - k(\cdot,y)\|_{L^1} \to 0$. Thus $K^*f \in C(\Omega)$. □

For general background on principal eigen-elements of positive operators and extensions of the Krein–Rutman theorem, we refer to [22].

Proposition 2.2 (Principal eigen-elements for L_m)**.** *Let $m \in (-\infty, 1)$. Then the following properties hold.*

1. *$s(L_m) < +\infty$,*
2. *If $s(L_m) > s(-\Gamma) = -\underline{\gamma}$, then there exist $f_m \in L^1_+(\Omega) \setminus \{0\}$ and $\Phi_m \in L^\infty_+(\Omega) \setminus \{0\}$ such that*

$$L_m f_m = s(L_m) f_m, \qquad L_m^* \Phi_m = s(L_m)\Phi_m.$$

Moreover,

$$\Phi_m \in C(\Omega) \qquad \text{and} \qquad \Phi_m(x) > 0 \quad \text{for all } x \in \Omega.$$

3. *Without assuming the spectral gap condition, the measure extension of L_m admits a non-zero positive Radon measure $\mu_m \in \mathcal{M}_+(\Omega) \setminus \{0\}$ associated with the same spectral value $s(L_m)$, where $s(L_m)$ is the spectral bound of L_m on $L^1(\Omega)$. In other words,*

$$L_m \mu_m = s(L_m)\mu_m$$

in the sense of measures.

Proof. First of all, we have the following inequality:

$$s(L_m) \le r(L_m) \le \|L_m\| \le (1-m)\overline{k} + \overline{\gamma} < +\infty$$

Assume now that $s(L_m) > s(-\Gamma) = -\underline{\gamma}$. By Lemma 2.1, the operator K is compact on $L^1(\Omega)$. Hence $L_m = (1-m)K - \Gamma$ is a compact perturbation of $-\Gamma$. Since $s(L_m)$ lies strictly above the spectral bound of the multiplication operator $-\Gamma$, Theorem 6.7 applies. Therefore $s(L_m)$ is a principal eigenvalue of L_m, and there exist $f_m \in L^1_+(\Omega) \setminus \{0\}, \Phi_m \in L^\infty_+(\Omega) \setminus \{0\}$, such that

$$L_m f_m = s(L_m) f_m, \qquad L_m^* \Phi_m = s(L_m)\Phi_m.$$

It remains to prove the additional regularity and strict positivity of Φ_m. Since $L_m^* = (1-m)K^* - \Gamma$, the eigenvalue equation gives $(1-m)K^*\Phi_m = (s(L_m)+\gamma)\Phi_m$. By Lemma 2.1, one has $K^*\Phi_m \in C(\Omega)$. Moreover, $s(L_m) > -\underline{\gamma}$ and $\gamma(x) \ge \underline{\gamma}$, thus we can define $\Phi_m(x) = \frac{(1-m)K^*\Phi_m(x)}{s(L_m)+\gamma(x)}$, and hence $\Phi_m \in C(\Omega)$. We now prove that Φ_m is strictly positive. Since $\Phi_m \ge 0$, $\Phi_m \not\equiv 0$, and $\Phi_m \in C(\Omega)$, there exist a nonempty open set $U \subset \Omega$ and a constant $c > 0$ such that $\Phi_m(x) \ge c$ for all $x \in U$. For every $y \in \Omega$, the eigenvalue equation gives

$$(s(L_m) + \gamma(y))\Phi_m(y) = (1-m)\int_\Omega k(x,y)\Phi_m(x)\,dx \ge (1-m)c\int_U k(x,y)\,dx.$$

By assumption (K3), the last integral is strictly positive for every $y \in \Omega$. Since $m < 1$, we obtain $(s(L_m) + \gamma(y))\Phi_m(y) > 0$. As $s(L_m) + \gamma(y) > 0$, it follows that $\Phi_m(y) > 0$ for all $y \in \Omega$.

We now turn to the case where no spectral gap is assumed. By the measure extension result stated in Theorem 6.8, the L^1-spectral value $s(L_m)$ admits a positive eigenmeasure for the measure extension of L_m. Thus there exists $\mu_m \in \mathcal{M}_+(\Omega) \setminus \{0\}$ such that $L_m\mu_m = s(L_m)\mu_m$ in the sense of measures. We emphasize that $s(L_m)$ still denotes the spectral bound of L_m on $L^1(\Omega)$. No uniqueness is asserted. □

We now establish explicit conditions on the transmission kernel and the recovery rate that govern the saturation of the spectral bound. Our findings characterize the threshold structure of the map $m \mapsto s(L_m)$ through two distinct mechanisms:

- **Spectral gap:** A regime where the spectral bound stays strictly isolated above the spectral bound, i.e., $s(L_m) > -\underline{\gamma}$.
- **Plateau formation:** A saturation regime where the spectral bound merges with the essential bound, i.e., $s(L_m) = -\underline{\gamma}$.

Proposition 2.3. *Let $A \subset \Omega$ be a compact subset with positive measure, and set*

$$a_A := \inf_{y \in A} \int_A \frac{k(x,y)}{\gamma(x) - \underline{\gamma}}\,dx.$$

For every $m \in (-\infty, 1)$ such that

$$m < \frac{a_A - 1}{a_A},$$

the spectral bound of L_m satisfies $s(L_m) > -\underline{\gamma}$.

Proof. The assumption is equivalent to $a_A > \frac{1}{1-m}$. For $\lambda > -\underline{\gamma}$, let $T_\lambda = (\lambda I + \Gamma)^{-1} K$. Then $\lambda \in \sigma(L_m) \Leftrightarrow 1 \in \sigma((1-m)T_\lambda)$, and $r(T_\lambda) = \frac{1}{1-m} \leftrightarrow \lambda = s(L_m)$. Define $H(\lambda, y) := \int_A \frac{k(x,y)}{\lambda + \gamma(x)}\, dx$. For every $y \in A$, $H(\lambda, y) \uparrow \int_A \frac{k(x,y)}{\gamma(x) - \underline{\gamma}}\, dx$ as $\lambda \downarrow -\underline{\gamma}$. Since A is compact and $H(\lambda, \cdot)$ is continuous, there exists $\lambda_0 > -\underline{\gamma}$ such that $\inf_{y \in A} H(\lambda_0, y) > \frac{1}{1-m}$. Consider the positive operator on $L^\infty(A)$, $(S_{\lambda_0}\varphi)(y) = \int_A \frac{k(x,y)}{\lambda_0 + \gamma(x)} \varphi(x)\, dx$. Then $S_{\lambda_0} \mathbf{1}_A \geq (\inf_{y \in A} H(\lambda_0, y))\, \mathbf{1}_A$, and therefore

$$r(S_{\lambda_0}) \geq \inf_{y \in A} H(\lambda_0, y) > \frac{1}{1-m}.$$

Moreover, S_{λ_0} is the compression of $T^*_{\lambda_0}$ to A, so positivity gives $r(S_{\lambda_0}) \leq r(T^*_{\lambda_0}) = r(T_{\lambda_0})$. Hence $r(T_{\lambda_0}) > \frac{1}{1-m}$. By the monotonicity of $\lambda \mapsto \rho(T_\lambda)$, it follows that

$$s(L_m) > \lambda_0 > -\underline{\gamma}.$$

□

Example 2.4.

Let the recovery rate and kernel be $\gamma(x) = 2 + x$ and $k(x, y) = 4x(1 + \frac{1}{2}\cos(2\pi y))$ on $\Omega = [0, 1]$. We compute the constant a analytically: $a = \inf_{y \in \Omega} \int_0^1 \frac{k(x,y)}{\gamma(x) - \underline{\gamma}}\, dx = 2$.

The condition for the existence of a spectral gap is $m < \frac{a-1}{a} = 0.5$. The numerical results shown in Figure 1 confirm that for $m < 0.5$, the spectral bound $s(L_m)$ (blue) stays above $-\underline{\gamma} = -2$ (red dashed).

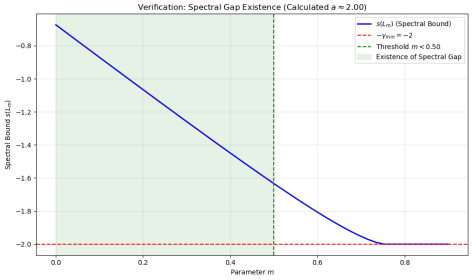


Figure 1: Numerical verification of the spectral gap.

Proposition 2.5. *Set*

$$b := \sup_{y \in \Omega} \int_\Omega \frac{k(x,y)}{\gamma(x) - \underline{\gamma}}\, dx.$$

For any $m \in [0, 1]$ satisfying

$$m > \frac{(b-1)}{b},$$

we have

$$s(L_m) = -\underline{\gamma}.$$

Proof. Assume for contradiction that $s(L_m) > -\underline{\gamma}$, and set $\delta := s(L_m) + \underline{\gamma} > 0$. By Theorem 6.7, there exists a positive eigenfunction $I \in L^1_+(\Omega) \setminus \{0\}$. The eigenvalue equation gives

$$I(x) = \frac{1 - m}{\delta + \gamma(x) - \underline{\gamma}} \int_\Omega k(x, y) I(y)\, dy.$$

Integrating over Ω and applying Fubini's theorem, we obtain

$$\|I\|_{L^1} = (1-m)\int_\Omega I(y)\left(\int_\Omega \frac{k(x,y)}{\delta+\gamma(x)-\underline{\gamma}}\,dx\right)dy \le (1-m)b\|I\|_{L^1},$$

where the last inequality follows from

$$\frac{1}{\delta+\gamma(x)-\underline{\gamma}} \le \frac{1}{\gamma(x)-\underline{\gamma}}$$

on $\{\gamma > \underline{\gamma}\}$, while the finiteness of b ensures that the contribution of $\{\gamma = \underline{\gamma}\}$ is zero.

Since $m > (b-1)/b$, one has $(1-m)b < 1$, which contradicts $I \not\equiv 0$. Hence $s(L_m) = -\underline{\gamma}$. □

Example 2.6.

Consider again the functions γ and k introduced in Example 2.4. Since $\underline{\gamma} = 2$,

$$b = \sup_{y\in[0,1]} \int_0^1 \frac{k(x,y)}{\gamma(x)-\underline{\gamma}}\,dx = 6.$$

Hence Proposition 2.5 yields

$$s(L_m) = -2 \qquad \text{for } m > \frac{5}{6}.$$

This bound is not sharp: using the separable structure of k, the actual transition occurs at $m_c = \frac{3}{4}$.

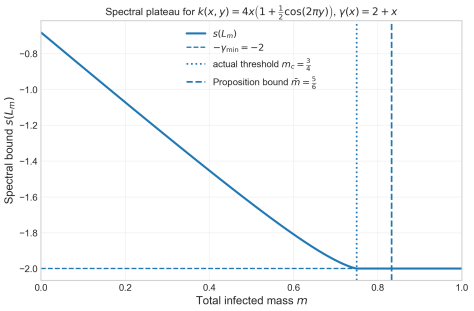


Figure 2: Spectral plateau. The bound 5/6 is sufficient but not sharp.

Proposition 2.7 (Uniqueness of the eigenfunction)**.** *Assume $\underline{\gamma} > 0$ and let $m \in [0,1)$ be such that $s(L_m) > s(-\Gamma)$. Let $J_1, J_2 \in L^1(\Omega)$ satisfy $L_m J_1 = s(L_m)J_1$ and $L_m J_2 = s(L_m)J_2$, $J_2 \not\equiv 0$. Then there exists $\alpha \in \mathbb{R}$ such that $J_1 = \alpha J_2$.*

Proof. Set $\widetilde{L}_m := L_m - s(L_m)\,\mathrm{Id}$. Then $s(\widetilde{L}_m) = 0$, and $L_m J = s(L_m)J \iff \widetilde{L}_m J = 0$. Moreover, $e^{t\widetilde{L}_m} = e^{-s(L_m)t}e^{tL_m}$, so $\widetilde{L}_m$ also generates a positive semigroup. Therefore, it is enough to prove the result in the case where the spectral bound is zero.

Thus, up to replacing L_m by $\widetilde{L}_m$, we may assume that $s(L_m) = 0$. We first observe that if $J \in L^1(\Omega)$ satisfies $L_m J = 0$, then $|J|$ is also an eigenfunction associated with the eigenvalue 0. Indeed, $J = (I - L_m)^{-1}J = R(1, L_m)J$, which is well defined since $s(L_m) = 0 < 1$.

Denoting $T := R(1, L_m)$, the positivity of T yields (see Chapter VI from [21]) $|J| = |TJ| \le T|J|$. Since $\underline{\gamma} > 0$, Proposition 2.2 give us a function $\Phi \in L^\infty_+(\Omega)$ which is strictly positive and satisfies $L_m^*\Phi = s(L_m)\Phi$. Furthermore, Φ verifies $\Phi = T^*\Phi$.

Applying the linear form $\langle \Phi, \cdot \rangle$ to the non-negative function $T|J| - |J|$ gives $\langle \Phi, T|J| - |J| \rangle = 0$. As $\Phi > 0$ almost everywhere, this implies $|J| = T|J|$ almost everywhere, hence $L_m|J| = 0$.

We now show that T is strongly positive. Let $f \in L^1_+(\Omega) \setminus \{0\}$ and set $u = Tf$. Positivity of the resolvent gives $u \ge 0$. Assume by contradiction that $S := \{x \in \Omega :\ u(x) = 0\}$ has positive measure. Since $f = (I - L_m)u = (1+\gamma)u - (1-m)Ku \ge 0$, we have

$$0 \le (1-m)\int_S \int_\Omega k(x,y)u(y)\,dy\,dx \le \int_S (1+\gamma(x))u(x)\,dx = 0.$$

Since $u = 0$ on S, this reduces to $\int_S \int_{S^c} k(x,y)u(y)\,dy\,dx = 0$. By Tonelli's theorem,

$$\int_{S^c} u(y) \left(\int_S k(x,y)\,dx \right) dy = 0.$$

Now $u(y) > 0$ for almost every $y \in S^c$, while $\int_S k(x,y)\,dx \geq 0$. Hence $\int_S k(x,y)\,dx = 0$ for a.e. $y \in S^c$, and therefore $\int_{S^c} \int_S k(x,y)\,dx\,dy = 0$.

Moreover, S^c has positive measure, since otherwise $u = 0$ a.e., which would imply $f = (I - L_m)u = 0$, contrary to $f \not\equiv 0$. Thus both S and S^c have positive measure, and the last equality contradicts Assumption (K2). Consequently, $u > 0$ a.e. on Ω.

As a consequence, any non-trivial solution J of $L_m J = 0$ has a constant sign. Indeed, if J changed sign, the function $v := |J| - J$ would satisfy $v \geq 0$, $v \not\equiv 0$, and $Tv = T|J| - TJ = |J| - J = v$. By strong positivity of T, this implies $v > 0$ almost everywhere, hence $|J| > J$ almost everywhere, which forces $J < 0$ almost everywhere. Then, we get the following result:

$$L_m J = 0 \qquad \Longrightarrow \qquad \begin{cases} J > 0 \quad \text{a.e} \\ \text{or} \\ J < 0 \quad \text{a.e} \end{cases}$$

Up to replacing J_i by $-J_i$, we may therefore assume that $J_1 > 0$ and $J_2 > 0$ almost everywhere. Define $w := \frac{J_1}{\|J_1\|_{L^1}} - \frac{J_2}{\|J_2\|_{L^1}}$. By linearity, $L_m w = 0$, so w has a constant sign almost everywhere. However, $\int_\Omega w(x)\,dx = 1 - 1 = 0$, and the only integrable function of constant sign with zero integral is the zero function. Hence $w \equiv 0$, which concludes the proof. □

2.2 The spectral curve $m \to s(L_m)$

Proposition 2.8. *The map $m \mapsto s(L_m)$ is continuous on $(-\infty, 1]$. Furthermore, there is $m_\star \leq 1$ such that the map is decreasing on $(-\infty, m_\star]$ and constant equal to $-\underline{\gamma}$ on $[m_\star, 1]$.*

Proof. Since $\|K\| \leq \overline{k}$ and Γ is bounded, $L_m = (1-m)K - \Gamma$ is bounded on $L^1(\Omega)$ and generates a positive semigroup for every $m \leq 1$. Let $m' < m$. Since $L_{m'} = L_m + (m - m')K \geq L_m$, Proposition 6.4(iii) gives $s(L_{m'}) \geq s(L_m)$. Thus $m \mapsto s(L_m)$ is non-increasing. Moreover, $L_1 = -\Gamma$, $s(L_1) = -\underline{\gamma}$, and therefore $s(L_m) \geq -\underline{\gamma}, m \leq 1$.

We now prove strict monotonicity above $-\underline{\gamma}$. Assume that $m' < m$ and $s(L_m) > -\underline{\gamma}$. By monotonicity, $s(L_{m'}) > -\underline{\gamma}$ as well. Proposition 2.2 yields a primal eigenfunction $f_{m'} > 0$ and a dual eigenfunction $\Phi_m > 0$ such that $L_{m'} f_{m'} = s(L_{m'}) f_{m'}$, $L_m^* \Phi_m = s(L_m)\Phi_m$. Using $L_{m'} = L_m + (m - m')K$, we obtain

$$\big(s(L_{m'}) - s(L_m)\big)\langle \Phi_m, f_{m'} \rangle = (m - m')\langle \Phi_m, K f_{m'} \rangle > 0.$$

Hence $s(L_{m'}) > s(L_m)$.

It remains to prove continuity. Let $m' < m < 1$. If $s(L_{m'}) = -\underline{\gamma}$, then monotonicity and the lower bound above imply directly $s(L_m) = s(L_{m'}) = -\underline{\gamma}$. Otherwise, Proposition 2.2 provides $f_{m'} > 0$, and $K f_{m'} = \frac{s(L_{m'}) + \gamma}{1 - m'} f_{m'}$. Therefore

$$\begin{aligned} L_m f_{m'} &= L_{m'} f_{m'} - (m - m')K f_{m'} \\ &\geq \left(s(L_{m'}) - (m - m')\frac{s(L_{m'}) + \overline{\gamma}}{1 - m'} \right) f_{m'}. \end{aligned}$$

Using the standard sub-eigenvector bound for positive semigroups, $Af \geq \lambda f$, $f > 0$, implies $s(A) \geq \lambda$, we deduce $0 \leq s(L_{m'}) - s(L_m) \leq (m - m')\frac{s(L_{m'})+\overline{\gamma}}{1-m'}$. This estimate gives both left- and right-continuity at every $m < 1$. Finally, for $m < 1$, $L_1 \leq L_m \leq -\underline{\gamma} I + (1-m)K$. Hence, again by Proposition 6.4(iii),

$$-\underline{\gamma} \leq s(L_m) \leq -\underline{\gamma} + (1-m)s(K) \leq -\underline{\gamma} + (1-m)\|K\|.$$

Letting $m \uparrow 1$ gives $s(L_m) \longrightarrow -\underline{\gamma} = s(L_1)$, so $m \mapsto s(L_m)$ is continuous on $(-\infty, 1]$. Set $m_\star := \inf\left\{m \leq 1 : \ s(L_m) = -\underline{\gamma}\right\}$. By continuity and monotonicity,

$$s(L_m) > -\underline{\gamma} \quad \text{for } m < m_\star, \qquad s(L_m) = -\underline{\gamma} \quad \text{for } m \geq m_\star.$$

The strict monotonicity proved above therefore shows that $m \mapsto s(L_m)$ is strictly decreasing on $(-\infty, m_\star]$ and constant equal to $-\underline{\gamma}$ on $[m_\star, 1]$. □

Remark 2.9. Assuming that $\underline{\gamma} = 0$, the graph of the function $m \mapsto s(L_m)$ can take one of three forms:

1. Zero Case: The spectral bound is constant equal to 0 for all $m \in [0, 1]$.

2. Plateau Case: The curve starts positive, reaches 0 at some $m_\star \in (0, 1)$, and remains constant equal to 0 on $[m_\star, 1]$.

3. Boundary Case: The curve is strictly decreasing and reaches 0 exactly at $m = 1$.

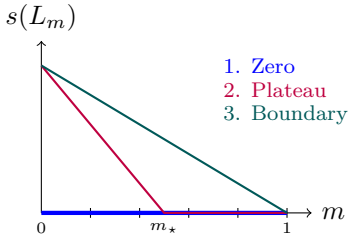


Figure 3: Three forms with $\underline{\gamma} = 0$

Remark 2.10. Assume that $\underline{\gamma} > 0$. The existence of an endemic equilibrium is completely determined by the zeros of the continuous and decreasing map $m \mapsto s(L_m)$. Since $s(L_1) = -\underline{\gamma} < 0$, we distinguish two cases based on the initial spectral bound $s(L_0)$:

Case $s(L_0) > 0$**:** The curve $s(L_m)$ must cross the horizontal axis at a unique value $m^* \in (0, 1)$ due to the Intermediate Value Theorem. This m^* corresponds to the unique endemic equilibrium.

Case $s(L_0) \leq 0$**:** Since the map is non-increasing starting from a non-positive value, $s(L_m)$ remains non-positive for all $m \in [0, 1]$. Thus, no endemic equilibrium exists.

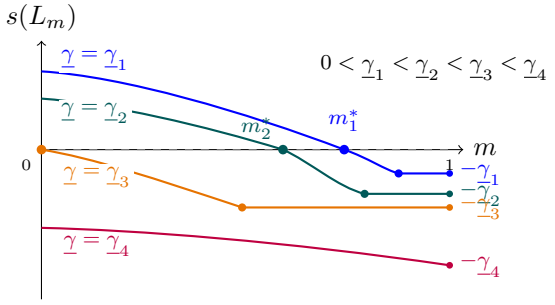


Figure 4: Behavior of $s(L_m)$ with spectral plateaus

A central contribution of this work is the identification of the epidemic threshold through the spectral bound $s(L_0)$ of the linearized operator. Rather than relying exclusively on the classical next-generation operator, we introduce a more general definition of the basic reproduction number, given by

$$\mathcal{R}_0 := \lim_{\lambda \downarrow 0} r(K(\lambda)), \qquad K(\lambda) := K(\lambda I + \Gamma)^{-1}.$$

This formulation is well suited to both regular and singular recovery regimes. In the case where the recovery rate is uniformly positive ($\underline{\gamma} > 0$), the operator Γ is invertible and the above definition coincides with the standard one, namely

$$\mathcal{R}_0 = r\Big(K\Gamma^{-1}\Big) = r(\mathcal{N}),$$

where $\mathcal{N} := K\Gamma^{-1}$ denotes the next-generation operator introduced by Diekmann et al. [19]. Within this classical setting, we recover the usual threshold property:

$$s(L_0) > 0 \quad (\text{resp. } = 0,\ < 0) \quad \Longleftrightarrow \quad \mathcal{R}_0 > 1 \quad (\text{resp. } = 1,\ < 1).$$

The proof is based on a refined spectral study of the family of positive operators $K(\lambda)$, inspired by the approach developed in [6], and combined with a resolvent decomposition of L_0.

From an epidemiological standpoint, the result shows that the spectral bound $s(L_0)$ can be interpreted as an intrinsic growth rate of the infection: if $\mathcal{R}_0 < 1$, the disease-free equilibrium is stable and the infection eventually disappears, whereas if $\mathcal{R}_0 > 1$, instability arises and the disease can persist. In this sense, the classical threshold principle known for ODE models naturally carries over to structured epidemic models involving nonlocal interactions.

Theorem 2.11. *Assume that $\underline{\gamma} > 0$. Then the following equivalences hold:*

$$s(L_0) \ > \ (=,\ <)\ 0 \quad \Longleftrightarrow \quad \mathcal{R}_0 \ > \ (=,\ <)\ 1.$$

Moreover, the spectral bound of L_0 satisfies the two-sided estimate

$$\max\left(u\ ;\ u.\mathbb{1}_{\mathcal{R}_0<1}\right) \ \leq \ s(L_0) \ \leq \ \min\left(v\ ;\ v.\mathbb{1}_{\mathcal{R}_0>1}\right).$$

Where we defined $u := \underline{\gamma}\,\mathcal{R}_0 - \bar{\gamma}$ and $v := \overline{\gamma}\mathcal{R}_0 - \underline{\gamma}$.

Proof. We refer to [37] for the first part of the theorem. It remains to establish the two-sided estimates. Since $\underline{\gamma} \leq \gamma(x) \leq \overline{\gamma}$, we have, in the order of positive operators,

$$K - \overline{\gamma} I \leq L_0 \leq K - \underline{\gamma} I.$$

By the monotonicity of the spectral bound for positive semigroups, we obtain $s(K-\overline{\gamma}I) \leq s(L_0) \leq s(K-\underline{\gamma}I)$. Since adding a scalar multiple of the identity shifts the spectrum, $s(K-\overline{\gamma}I) = s(K) - \overline{\gamma}$, $s(K-\underline{\gamma}I) = s(K) - \underline{\gamma}$. Moreover, because K is a positive bounded operator on a Banach lattice, its spectral bound coincides with its spectral radius: $s(K) = r(K)$, see for instance Schaefer [35]. Hence

$$r(K) - \overline{\gamma} \leq s(L_0) \leq r(K) - \underline{\gamma}.$$

On the other hand,

$$\frac{1}{\overline{\gamma}} I \leq \Gamma^{-1} \leq \frac{1}{\underline{\gamma}} I.$$

Since K is positive, $\frac{1}{\overline{\gamma}} K \leq K\Gamma^{-1} \leq \frac{1}{\underline{\gamma}} K$. Using again the monotonicity of the spectral radius for positive operators, we get $\frac{1}{\overline{\gamma}} r(K) \leq r(K\Gamma^{-1}) \leq \frac{1}{\underline{\gamma}} r(K)$. Since $\mathcal{R}_0 = r(K\Gamma^{-1})$, this gives $\underline{\gamma}\,\mathcal{R}_0 \leq r(K) \leq \overline{\gamma}\,\mathcal{R}_0$. Combining this with the previous estimate yields $u := \underline{\gamma}\,\mathcal{R}_0 - \overline{\gamma} \leq s(L_0) \leq \overline{\gamma}\,\mathcal{R}_0 - \underline{\gamma} =: v$.

Finally, the threshold equivalence gives the additional information

$$s(L_0) < 0 \quad \text{if } \mathcal{R}_0 < 1, \qquad s(L_0) > 0 \quad \text{if } \mathcal{R}_0 > 1.$$

Thus we may refine the previous bounds as

$$\max\left(u,\ u\,\mathbb{1}_{\mathcal{R}_0<1}\right) \le s(L_0) \le \min\left(v,\ v\,\mathbb{1}_{\mathcal{R}_0>1}\right).$$

□

As already mentioned in the introduction, the equivalence between the spectral bound $s(L_0)$ and the basic reproduction number R_0 is not new in the regular setting. It is a direct consequence of the general spectral framework developed by H.Thieme in [37], which establishes the correspondence between the sign of the spectral bound and the threshold condition $R_0 \gtrless 1$.

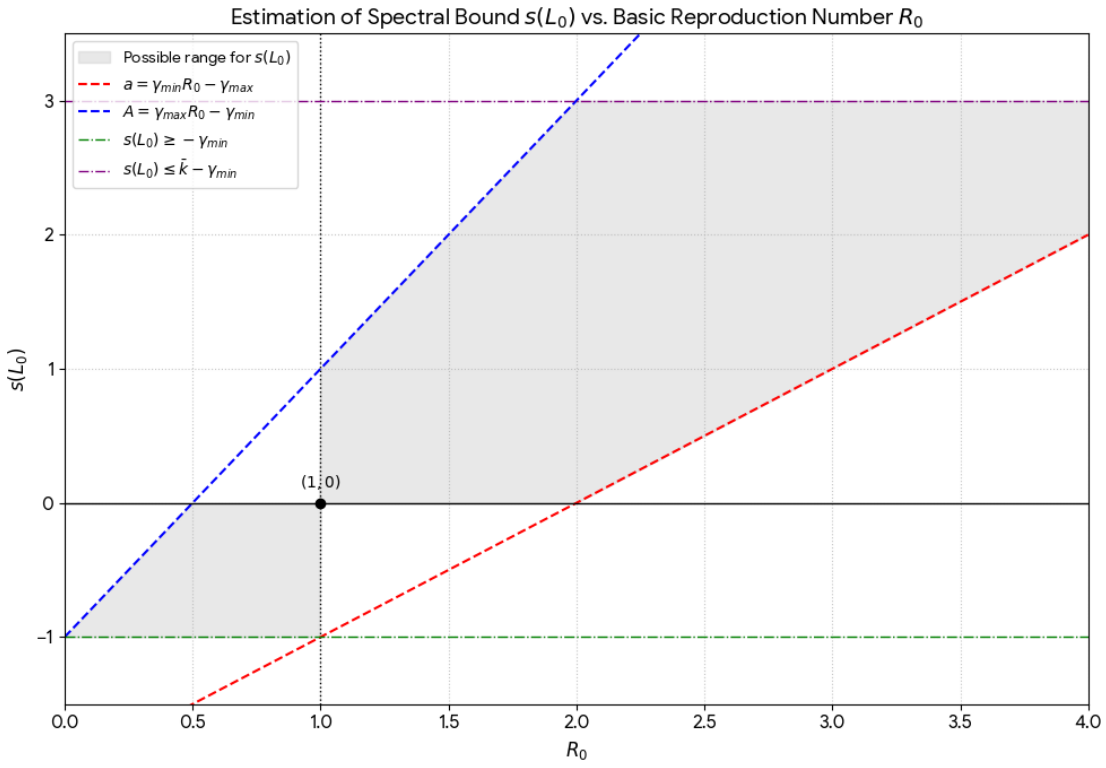


Figure 5: Admissible values of the spectral bound $s(L_0)$ as a function of the reproduction number $\mathcal{R}_0$. The shaded regions indicate the possible locations of $s(L_0)$ under the two-sided estimates of Theorem 2.11. The vertical line $\mathcal{R}_0 = 1$ marks the epidemic threshold, at which $s(L_0) = 0$.

Figure 5 provides a graphical illustration of Theorem 2.11. The horizontal axis represents the reproduction number $\mathcal{R}_0$, while the vertical axis corresponds to the spectral bound $s(L_0)$ of the linearized operator.

The admissible region is constrained by the a priori bounds $-\underline{\gamma} \le s(L_0) \le \overline{k} - \underline{\gamma}$ and $\mathcal{R}_0 \le \overline{k}/\underline{\gamma}$. The dashed affine functions $u(r) = \underline{\gamma} r - \overline{\gamma}$ and $v(r) = \overline{\gamma} r - \underline{\gamma}$ define lower and upper estimates for $s(L_0)$.

In accordance with Theorem 2.11, the sign of the spectral bound is entirely determined by the position of $\mathcal{R}_0$ with respect to the threshold value 1: for $\mathcal{R}_0 < 1$, one has $s(L_0) < 0$, whereas for $\mathcal{R}_0 > 1$, the spectral bound becomes positive. The critical case $\mathcal{R}_0 = 1$ corresponds to $s(L_0) = 0$.

Remark 2.12. The conclusion of Theorem 2.11 may fail when $\underline{\gamma} = 0$. To see this, consider the kernel $k(x, y) = y^2$ and the loss $\gamma(x) = 2x^2$. For an eigenpair of $L_0 = K - \Gamma$ with eigenvalue λ one must have (setting $c = \int_\Omega y^2 f(y)\, dy \neq 0$)

$$f(x) = \tfrac{c}{\lambda+2x^2}, \qquad c = \int_\Omega \tfrac{cy^2}{\lambda+2y^2}\, dy.$$

If $c = 0$ that implies $f = 0$ which contradicts the definition of (λ, f). So we can divide by c to obtain

$$\int_\Omega \tfrac{y^2}{\lambda+2y^2}\, dy = 1.$$

For real $\lambda > 0$, the eigenvalue condition is therefore equivalent to

$$\Phi(\lambda) := \int_0^1 \frac{y^2}{\lambda + 2y^2}\, dy = 1.$$

Set $\Omega = [0, 1]$, then

$$\Phi(0) = \int_0^1 \tfrac{y^2}{2y^2}\, dy = \tfrac{1}{2} < 1,$$

and $\Phi(\lambda)$ is strictly decreasing for $\lambda > 0$, hence there is no solution $\lambda > 0$ of $\Phi(\lambda) = 1$. The spectral bound is therefore not positive and by $s(L_0) \geq s(-\Gamma) = 0$ we finally get $s(L_0) = 0$.

On the other hand the Next-Generation operator is defined as follows :

$$\mathcal{N} f(x) = \int_0^1 \tfrac{f(y)}{2} dy.$$

It's easy to see $r(\mathcal{N}) = \frac{1}{2}$. Thus $r(\mathcal{N}) = 1/2$ while $s(L_0) = 0$, showing that the classical next-generation characterization of the critical threshold fails in the singular case.

When the recovery rate vanishes locally (i.e. $\underline{\gamma} = 0$), the basic reproduction number $\mathcal{R}_0$ may become infinite, and the conventional next-generation operator $\mathcal{N} = K\Gamma^{-1}$ is no longer adequately defined in the standard framework of Diekmann et al. [19]. Consequently, the usual threshold characterization based on $\mathcal{R}_0$ fails. The following result tackles this challenge by extending the connection between the spectral bound $s(L_0)$ and epidemic invasion to this singular regime.

Theorem 2.13. *Assume that $\underline{\gamma} = 0$ and $\gamma(\cdot) \not\equiv 0$. Then the following equivalences hold:*

$$s(L_0) \; > \; 0 \quad \Longleftrightarrow \quad \mathcal{R}_0 \; > \; 1.$$

Moreover the spectral bound of L_0 satisfied the two-sided estimate

$$0 \leq s(L_0) \leq \left(\overline{\gamma}\mathcal{R}_0 - \underline{\gamma}\right) \mathbb{1}_{\mathcal{R}_0 > 1}$$

As a consequence we have $\mathcal{R}_0 \leq 1 \Leftrightarrow s(L_0) = 0$.

Proof. For every $\lambda > 0$, we have the factorization

$$\lambda I - L_0 = \big(I - \mathcal{K}(\lambda)\big)(\lambda I + \Gamma), \qquad \mathcal{K}(\lambda) := K(\lambda I + \Gamma)^{-1}.$$

Hence

$$\lambda \in \sigma(L_0) \quad \Longleftrightarrow \quad 1 \in \sigma(\mathcal{K}(\lambda)).$$

Moreover, $\lambda \mapsto \mathcal{K}(\lambda)$ is decreasing and continuous on $(0, +\infty)$, so $\lambda \mapsto r(\mathcal{K}(\lambda))$ is non-increasing and continuous, with $r(\mathcal{K}(\lambda)) \to 0$ as $\lambda \to +\infty$.

Assume first that $\mathcal{R}_0 > 1$. Then $r(\mathcal{K}(\lambda_0)) > 1$ for some $\lambda_0 > 0$, and by continuity there exists $\lambda^* > 0$ such that $r(\mathcal{K}(\lambda^*)) = 1$. Since $\mathcal{K}(\lambda^*)$ is positive and compact, Krein–Rutman gives $1 \in \sigma(\mathcal{K}(\lambda^*))$. Hence $\lambda^* \in \sigma(L_0)$ and therefore $s(L_0) > 0$.

Conversely, assume that $s(L_0) > 0$. By [21, Chapter VI, Theorem 1.10], $s(L_0) \in \sigma(L_0)$, hence $1 \in \sigma(\mathcal{K}(s(L_0)))$ and $r(\mathcal{K}(s(L_0))) \geq 1$. By monotonicity,

$$\mathcal{R}_0 = \lim_{\lambda \downarrow 0} r(\mathcal{K}(\lambda)) \geq r(\mathcal{K}(s(L_0))) \geq 1.$$

We now exclude $\mathcal{R}_0 = 1$. In that case, $r(\mathcal{K}(\lambda)) = 1$ for every $\lambda \in (0, s(L_0)]$. Choose $0 < \lambda_1 < \lambda_2 \leq s(L_0)$. By Krein–Rutman and irreducibility, there exist $u_1 \in L^1_+(\Omega)\setminus\{0\}$ and $\phi_2 \in L^\infty_+(\Omega)\setminus\{0\}$ such that

$$\mathcal{K}(\lambda_1)u_1 = u_1, \qquad \mathcal{K}(\lambda_2)^*\phi_2 = \phi_2,$$

with $u_1 > 0$ a.e. Since $K^*\phi_2 = (\lambda_2 + \gamma)\phi_2$, we obtain

$$\langle \phi_2, (\mathcal{K}(\lambda_1) - \mathcal{K}(\lambda_2))u_1 \rangle = (\lambda_2 - \lambda_1) \int_\Omega \frac{\phi_2(y)u_1(y)}{\lambda_1 + \gamma(y)}\, dy > 0.$$

On the other hand, the two eigenvalue relations give

$$\langle \phi_2, (\mathcal{K}(\lambda_1) - \mathcal{K}(\lambda_2))u_1 \rangle = \langle \phi_2, u_1 \rangle - \langle \phi_2, u_1 \rangle = 0,$$

a contradiction. Therefore $\mathcal{R}_0 > 1$, and thus

$$s(L_0) > 0 \quad \Longleftrightarrow \quad \mathcal{R}_0 > 1.$$

Finally, since $\underline{\gamma} = 0$, one has $s(-\Gamma) = 0$, and $L_0 = K - \Gamma \geq -\Gamma$. Hence $s(L_0) \geq 0$, so the previous equivalence yields

$$\mathcal{R}_0 \leq 1 \quad \Longleftrightarrow \quad s(L_0) = 0.$$

The two-sided estimate follows from the same comparison argument as in Theorem 2.11, with $\underline{\gamma} = 0$. □

Remark 2.14 (Comparison with the classical next-generation operator). The generalized definition of $\mathcal{R}_0$ may remain meaningful even when the classical next-generation operator $K\Gamma^{-1}$ is not available as a bounded operator on $L^1(\Omega)$. Indeed, consider

$$\Omega = [0, 1], \qquad \gamma(x) = x, \qquad k(x, y) = \frac{x}{x + y}.$$

As shown in Appendix Proposition 6.9,

$$\mathcal{R}_0 = \lim_{\lambda \downarrow 0} r\Big(K(\lambda I + \Gamma)^{-1}\Big) = \pi.$$

On the other hand, the classical next-generation operator $\mathcal{N} = K\Gamma^{-1}$ does not define a bounded operator on $L^1(0,1)$, since $\mathbf{1} \notin D(\Gamma^{-1})$ and

$$\Gamma^{-1}\mathbf{1}(x) = \frac{1}{x} \notin L^1(0,1).$$

In fact, when $\mathcal{N}$ is considered on its natural domain

$$D(\mathcal{N}) = \left\{ f \in L^1(0,1) : \frac{f}{x} \in L^1(0,1) \right\},$$

its resolvent set is empty. Indeed, for every $f \in D(\mathcal{N})$, the function $\mathcal{N}f$ admits a continuous representative on $[0,1]$ which vanishes at $x = 0$. Hence the constant function $\mathbf{1}$ cannot belong to the range of $\mathcal{N} - \lambda I$ for any $\lambda \in \mathbb{C}$: for $\lambda \neq 0$, such an equation would force $f(x) \to -1/\lambda$ as $x \downarrow 0$, contradicting $f/x \in L^1(0,1)$. Therefore $\sigma(\mathcal{N}) = \mathbb{C}$, and in particular

$$\sup_{\lambda \in \sigma(\mathcal{N})} |\lambda| = +\infty.$$

Thus the regularized reproduction number remains finite even though the classical next-generation operator has no resolvent point. Moreover, for the rescaled kernels

$$k_c(x,y) = c\,k(x,y), \qquad 0 < c < \frac{1}{\pi},$$

one has $\mathcal{R}_0 = c\pi < 1$, whereas the corresponding next-generation operator still satisfies $\sigma(\mathcal{N}_c) = \mathbb{C}$.

2.3 Endemic equilibrium and endemic measures

Proof of the Theorem 1.6. Let μ be an endemic measure with mass m^*. Since $x \mapsto k(x,y)$ belongs to $L^1(\Omega)$ for every $y \in \Omega$, we can define the nonnegative measurable function

$$\rho_\mu(x) := \int_\Omega k(x,y)\,\mu(dy).$$

By Tonelli's theorem, $\int_\Omega \rho_\mu(x)\,dx = \int_\Omega \int_\Omega k(x,y)\,dx\,\mu(dy)$. Moreover, since $\sup_{y\in\Omega} \int_\Omega k(x,y)\,dx$ is finite by assumption $(K1)$, and since μ is a finite Radon measure, we obtain $\int_\Omega \rho_\mu(x)\,dx \leq \left(\sup_{y\in\Omega} \int_\Omega k(x,y)\,dx\right) \mu(\Omega)$. Thus $\rho_\mu \in L^1(\Omega)$.

Now, for every Borel set $A \subset \Omega$, the measure $K\mu$ satisfies

$$(K\mu)(A) = \int_\Omega \int_A k(x,y)\,dx\,\mu(dy) = \int_A \rho_\mu(x)\,dx.$$

Therefore $K\mu$ is absolutely continuous with respect to the Lebesgue measure, with density ρ_μ: $K\mu = \rho_\mu(x)\,dx$. Using the stationary relation $(1-m^*)K\mu = \Gamma\mu$, we deduce that $\Gamma\mu$ is also absolutely continuous. Restricting to $\Omega \setminus Z = \{x \in \Omega : \gamma(x) > 0\}$, this yields that $\mu_{|\Omega\setminus Z}$ is absolutely continuous, with density given by

$$I_{|\Omega\setminus Z}(x) = \tfrac{1-m^*}{\gamma(x)} \int_\Omega k(x,y)\,\mu(dy).$$

Setting $m_{\text{out}} := \mu(\Omega \setminus Z)$, we obtain

$$m_{\text{out}} = (1-m^*) \int_\Omega \left(\int_{\Omega\setminus Z} \tfrac{k(x,y)}{\gamma(x)}\,dx \right) \mu(dy) = (1-m^*) \int_\Omega \mathcal{B}(y)\,\mu(dy).$$

Let $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sing}}$ denote the Lebesgue decomposition of μ. Since $\Gamma\mu$ is absolutely continuous and $\Gamma\mu = \gamma\mu_{\mathrm{ac}} + \gamma\mu_{\mathrm{sing}}$, the uniqueness of this decomposition implies $\gamma\mu_{\mathrm{sing}} = 0$.

For any compact set $A \subset \Omega \setminus Z$, there exists $c > 0$ such that $\gamma \geq c$ on A, hence $0 = (\gamma\mu_{\mathrm{sing}})(A) \geq c\,\mu_{\mathrm{sing}}(A)$, which gives $\mu_{\mathrm{sing}}(A) = 0$. By inner regularity, it follows that $\mu_{\mathrm{sing}}(\Omega \setminus Z) = 0$, and therefore $\mathrm{supp}(\mu_{\mathrm{sing}}) \subset Z$.

Assume that $\mathcal{B}(y) = +\infty$ for all $y \in Z$. If $\mu(Z) > 0$, then $\int_Z \mathcal{B}(y)\,\mu(dy) = +\infty$, which implies $m_{\mathrm{out}} = +\infty$, contradicting $m_{\mathrm{out}} \leq \mu(\Omega) = m^* < 1$. Thus $\mu(Z) = 0$.

Suppose $0 < \overline{\mathcal{B}} < +\infty$. Since $\mathcal{B}(y) \leq \overline{\mathcal{B}}$ on Ω, we have $m_{\mathrm{out}} \leq (1 - m^*)\overline{\mathcal{B}}\,m^*$. If $\mu(Z) = 0$, then $m_{\mathrm{out}} = m^*$, and thus $1 \leq (1-m^*)\overline{\mathcal{B}}$, that is $m^* \leq \frac{\overline{\mathcal{B}}-1}{\overline{\mathcal{B}}}$, which contradicts the hypothesis. Therefore $\mu(Z) > 0$.

Finally, if $|Z| = 0$, then $\mu_{\mathrm{ac}}(Z) = 0$, so $\mu(Z) > 0$ implies $\mu_{\mathrm{sing}}(Z) > 0$, and in particular $\mu_{\mathrm{sing}} \neq 0$. □

Remark 2.15 (A non-empty critical plateau with $\mathcal{B} = +\infty$ on Z). Shows that the concentration result in Theorem 1.6 is not purely formal. For the example introduced in Remark 2.14,

$$\Omega = [0,1], \qquad \gamma(x) = x, \qquad k(x,y) = \frac{x}{x+y},$$

one has $Z = \{0\}$ and

$$\mathcal{B}(0) = \int_0^1 \frac{k(x,0)}{\gamma(x)}\,dx = \int_0^1 \frac{1}{x}\,dx = +\infty.$$

Hence the no-mass condition of Theorem 1.6 applies, and

$$\mu(Z) = 0$$

for every endemic measure μ with $\mu(\Omega) \in (0,1)$.

On the other hand, Appendix 6.9 shows that

$$s(L_m) = 0 \qquad \text{for all} \qquad m \in \left[1 - \frac{1}{\pi}, 1\right),$$

whereas

$$s(L_m) > 0 \qquad \text{if and only if} \qquad m < 1 - \frac{1}{\pi}.$$

Hence the plateau starts at the strictly subunit mass

$$m_\star = 1 - \frac{1}{\pi} \in (0,1).$$

Thus, although the generalized reproduction number is finite, $\mathcal{R}_0 = \pi$, and gives rise to a nontrivial critical plateau, the divergence of $\mathcal{B}$ on Z prevents any endemic equilibrium measure from charging the vanishing recovery set.

Proposition 2.16 (Strict positivity of the spectral bound below an equilibrium)**.** *Assume that* $\underline{\gamma} = 0$ *and that the following additional assumptions:*

(H1) There exists $\varepsilon > 0$ *such that*

$$\inf_{x \in \Omega} \int_{\Omega_\varepsilon} k(x,y)\,dy > 0, \qquad \Omega_\varepsilon := \{y \in \Omega : \gamma(y) \geq \varepsilon\}.$$

(H2) $|Z| = 0$.

(H3) $x \mapsto k(x,\cdot) \in \mathcal{C}(\Omega, L^\infty(\Omega))$.

Let $m \in [0,1)$ and suppose there exists $I^* \in L^1_+(\Omega) \setminus \{0\}$ such that

$$L_m I^* = 0.$$

Then for all $m' < m$, one has

$$s(L_{m'}) > 0.$$

Proof. From $L_m I^* = 0$, we have $(1-m)KI^* = \gamma I^*$. By $(H3)$, $KI^* \in C(\Omega)$, hence $\gamma I^* \in C(\Omega)$. Fix $\varepsilon > 0$ as in $(H1)$. On $\Omega_\varepsilon = \{x : \gamma(x) \geq \varepsilon\}$,

$$I^*(x) = \tfrac{1-m}{\gamma(x)} KI^*(x),$$

so I^* is continuous on Ω_ε. We first claim that $I^* > 0$ a.e. on Ω. Indeed, let $A := \{x \in \Omega : I^*(x) = 0\}$. If $|A| > 0$, then, since $I^* \not\equiv 0$, one also has $|\Omega \setminus A| > 0$. From $(1-m)KI^* = \gamma I^*$, we have $KI^* = 0$ a.e. on A, and therefore

$$0 = \int_A KI^*(x)\,dx = \int_{\Omega\setminus A} I^*(y) \left(\int_A k(x,y)\,dx \right) dy.$$

Since $I^* > 0$ a.e. on $\Omega \setminus A$, it follows that

$$\int_A k(x,y)\,dx = 0 \qquad \text{for a.e. } y \in \Omega \setminus A,$$

which contradicts assumption (K2) applied to $\Omega \setminus A$. Hence $I^* > 0$ a.e. on Ω.

Now, by (H3), $KI^* \in C(\Omega)$. Moreover, for every $x \in \Omega_\varepsilon$,

$$KI^*(x) \geq \int_{\Omega_\varepsilon} k(x,y) I^*(y)\,dy > 0,$$

where the last inequality follows from (H1) and $I^* > 0$ a.e. on Ω_ε. Consequently,

$$I^*(x) = \frac{1-m}{\gamma(x)} KI^*(x) > 0 \qquad \text{on } \Omega_\varepsilon.$$

Since I^* is continuous on the compact set Ω_ε, we obtain

$$a_\varepsilon := \min_{\Omega_\varepsilon} I^* > 0.$$

Let $0 < \eta < \varepsilon$. Define $u_\eta := \mathbf{1}_{\Omega_\eta} I^*$, $v_\eta := \mathbf{1}_{\Omega_\eta^c} I^*$, so that $I^* = u_\eta + v_\eta$. Since $L_m I^* = 0$ and $I^* = u_\eta + v_\eta$, we have $(1-m)K(u_\eta + v_\eta) = \Gamma u_\eta + \Gamma v_\eta$. Therefore

$$\Gamma u_\eta = (1-m)Ku_\eta + (1-m)Kv_\eta - \Gamma v_\eta.$$

Let $m' < m$, using the definition of $L_{m'}$, we obtain

$$\begin{aligned} L_{m'} u_\eta &= (1-m')Ku_\eta - \Gamma u_\eta \\ &= (m-m')Ku_\eta - (1-m)Kv_\eta + \Gamma v_\eta. \end{aligned}$$

On Ω_η, one has $v_\eta = 0$, hence $\Gamma v_\eta = 0$. Consequently,

$$L_{m'} u_\eta = (m - m')K u_\eta - (1 - m)K v_\eta \qquad \text{a.e. on } \Omega_\eta.$$

For all $x \in \Omega$,

$$K u_\eta(x) \geq \int_{\Omega_\varepsilon} k(x, y) I^*(y)\, dy \geq a_\varepsilon \int_{\Omega_\varepsilon} k(x, y)\, dy,$$

hence $K u_\eta(x) \geq a_\varepsilon c_\varepsilon$ where $c_\varepsilon := \inf_{x \in \Omega} \int_{\Omega_\varepsilon} k(x, y) dy > 0$ by (H1).

Let $R > 0$ and define $I_R := \min(I^*, R)$. Then

$$K v_\eta = K(\mathbf{1}_{\Omega_\eta^c}(I^* - I_R)) + K(\mathbf{1}_{\Omega_\eta^c} I_R).$$

Using $k \in L^\infty$, we have $\|K(I^* - I_R)\|_{L^\infty} \leq \|k\|_{L^\infty} \|I^* - I_R\|_{L^1} \xrightarrow[R \to +\infty]{} 0$, and $\|K(\mathbf{1}_{\Omega_\eta^c} I_R)\|_{L^\infty} \leq R\|k\|_{L^\infty} |\Omega_\eta^c|$. Since $|\Omega_\eta^c| \xrightarrow[\eta \to 0]{} 0$ by (H2), we obtain $\|K v_\eta\|_{L^\infty} \xrightarrow[\eta \to 0]{} 0$.

Choose η small so that $(1-m)\|K v_\eta\|_{L^\infty} \leq \frac{1}{2}(m-m')a_\varepsilon c_\varepsilon$. Then $L_{m'} u_\eta \geq \frac{1}{2}(m-m')a_\varepsilon c_\varepsilon \quad$ on $\quad \Omega_\eta$, and $L_{m'} u_\eta \geq 0$ elsewhere. Hence

$$L_{m'} u_\eta \geq \eta_\varepsilon \mathbf{1}_{\Omega_\eta}, \qquad \eta_\varepsilon := \tfrac{1}{2}(m - m')a_\varepsilon c_\varepsilon.$$

Since $\gamma \geq \eta$ on Ω_η, $u_\eta(x) \leq \frac{1-m}{\eta}\|K I^*\|_{L^\infty}$, so $u_\eta \leq M \mathbf{1}_{\Omega_\eta}$ with $M := \frac{1-m}{\eta}\|K I^*\|_{L^\infty}$. Thus

$$L_{m'} u_\eta \geq \alpha u_\eta, \qquad \alpha := \tfrac{\eta_\varepsilon}{M} > 0.$$

Assume by contradiction that $s(L_{m'}) < \alpha$. Then $(\alpha I - L_{m'})$ is invertible and its resolvent $R(\alpha, L_{m'}) = (\alpha I - L_{m'})^{-1}$ is a positive operator. From $L_{m'} u_\eta \geq \alpha u_\eta$, we have $(\alpha I - L_{m'}) u_\eta \leq 0$. Applying the positive operator $R(\alpha, L_{m'})$ yields $u_\eta \leq 0$, which contradicts $u_\eta \geq a_\varepsilon \mathbf{1}_{\Omega_\varepsilon} > 0$ on the set Ω_η.

Therefore

$$s(L_{m'}) \geq \alpha > 0.$$

□

Corollary 2.17 (Singular eigenmeasures on the spectral plateau)**.** *Assume the hypotheses of Proposition 2.16. Recall* $m_\star = \min\{m \in (-\infty, 1] \ : \ \forall\, 1 \geq m' \geq m,\, s(L_{m'}) = 0\}$. *Then for every* $m \in (m_\star, 1]$, *any positive eigenmeasure* $\mu_m \in \mathcal{M}_+(\Omega) \setminus \{0\}$ *satisfying* $L_m \mu_m = 0$ *has a non-zero singular part with respect to the Lebesgue measure.*

Proof. Let $m \in (m_\star, 1)$, and let $\mu_m \in \mathcal{M}_+(\Omega) \setminus \{0\}$ satisfy $L_m \mu_m = 0$. Assume by contradiction that μ_m is absolutely continuous with respect to the Lebesgue measure. Then there exists $I_m \in L^1_+(\Omega) \setminus \{0\}$ such that

$$\mu_m(dx) = I_m(x)\, dx.$$

Since $L_m \mu_m = 0$ in the sense of measures, we obtain $L_m I_m = 0$ in $L^1(\Omega)$.

By Proposition 2.16, applied to the non-zero nonnegative eigenfunction I_m, we have $s(L_{m'}) > 0$ for every $m' < m$. Since $m_\star < m$, we may take $m' = m_\star$, and we obtain

$$s(L_{m_\star}) > 0.$$

However, by the definition of $m_\star$ recalled in the statement, the infimum is attained and $s(L_{m_\star}) = 0$. This is a contradiction. Finally, the case $m = 1$ is trivial. □

Remark 2.18 (Shift by the bottom of the recovery rate). The two previous results have been stated in the singular case $\underline{\gamma} = 0$, in order to simplify the notation and to emphasize the relation with the plateau $s(L_m) = 0$. However, the same arguments apply when $\underline{\gamma} > 0$, after shifting the recovery rate by its minimum.

More precisely, one replaces γ by

$$\widetilde{\gamma} := \gamma - \underline{\gamma},$$

and studies

$$\widetilde{L}_m := L_m + \underline{\gamma} I = (1-m)K - (\gamma - \underline{\gamma}).$$

Then

$$s(\widetilde{L}_m) = s(L_m) + \underline{\gamma}.$$

Therefore the level 0 for $\widetilde{L}_m$ corresponds exactly to the level $-\underline{\gamma}$ for L_m. In particular, the preceding arguments remain valid with the substitutions

$$\gamma \quad \rightsquigarrow \quad \gamma - \underline{\gamma}, \qquad s(L_m) = 0 \quad \rightsquigarrow \quad s(L_m) = -\underline{\gamma}.$$

Thus the same conclusions hold for the plateau at the bottom of the essential spectrum.

Definition 2.19 (Incoming flux into the zero-recovery set)**.** We define the incoming flux coefficient towards Z by

$$C_Z(y) := \int_Z k(x,y)\,dx, \qquad y \in \Omega.$$

Proposition 2.20 (No incoming flux on the zero-recovery set)**.** *Let μ be an endemic measure with mass $m := \mu(\Omega) \in (0,1)$. Then $\rho_\mu = 0$ a.e. on Z, or equivalently,*

$$\int_\Omega C_Z(y)\,\mu(dy) = 0.$$

In particular,

$$\mu(\{C_Z > 0\}) = 0.$$

Consequently, if $C_Z > 0$ on Ω, then no endemic measure with mass in $(0,1)$ exists. In particular, this conclusion holds whenever Z contains a nonempty open subset because of Assumption $(K3)$.

Proof. Recall $\rho_\mu(x) := \int_\Omega k(x,y)\,\mu(dy)$. Since μ is an endemic measure, it satisfies $(1-m)K\mu = \Gamma\mu$ in the sense of measures. By definition of ρ_μ, this reads

$$(1-m)\rho_\mu(x)\,dx = \gamma(x)\,\mu(dx).$$

Let $A \subset Z$ be Borel. Since $\gamma = 0$ on Z, we obtain $(1-m)\int_A \rho_\mu(x)\,dx = \int_A \gamma(x)\,\mu(dx) = 0$. As $m < 1$, this gives $\int_A \rho_\mu(x)\,dx = 0$ for every Borel set $A \subset Z$. Hence $\rho_\mu = 0$ a.e. on Z.

Integrating over Z and using Tonelli's theorem, we get

$$0 = \int_Z \rho_\mu(x)\,dx = \int_Z \int_\Omega k(x,y)\,\mu(dy)\,dx = \int_\Omega C_Z(y)\,\mu(dy).$$

Since $C_Z \geq 0$, it follows that $C_Z = 0$ μ-almost everywhere, that is, $\mu(\{C_Z > 0\}) = 0$.

If $C_Z > 0$ on Ω, then the previous identity forces $\mu(\Omega) = 0$, which contradicts $m > 0$. Hence no endemic measure with mass in $(0, 1)$ exists.

Finally, assume that Z contains a nonempty open subset A. By assumption (K3), $\int_A k(x, y)\,dx > 0$ for every $y \in \Omega$. Since $A \subset Z$, we have

$$C_Z(y) = \int_Z k(x, y)\,dx \geq \int_A k(x, y)\,dx > 0 \quad \text{for every } y \in \Omega.$$

Thus $C_Z > 0$ on Ω, and the conclusion follows from the previous paragraph. □

Remark 2.21. The previous proposition does not imply that the absolutely continuous part of μ vanishes on Z. If $|Z| > 0$, the absolutely continuous part may still charge Z. What the stationarity equation imposes is instead the absence of incoming infectious flux into the zero-recovery region:

$$K\mu = 0 \quad \text{a.e. on } Z.$$

Equivalently, the equilibrium measure can only charge strains y for which the total transmission towards Z vanishes, namely

$$C_Z(y) = \int_Z k(x, y)\,dx = 0.$$

Proposition 2.22 (Explicit endemic measures in the singular House-of-Cards case)**.** *Assume that the kernel is of House-of-Cards type,*

$$k(x, y) = Q(x),$$

with $Q \geq 0$, and that

$$\gamma \in C(\Omega), \qquad \gamma \geq 0, \qquad Z = \{0\}.$$

Assume moreover that

$$1 < \rho := \int_{\Omega\setminus\{0\}} \frac{Q(x)}{\gamma(x)}\,dx < +\infty.$$

Then $m_\star = 1 - \frac{1}{\rho}$, and, for every $m \in [m_\star, 1]$, the endemic equilibrium measure with total mass m is uniquely given by

$$\mu_m(dx) = m(1 - m)\frac{Q(x)}{\gamma(x)}\,dx + m(1 - \rho(1 - m))\delta_0(dx).$$

Proof. Let $\mu \in \mathcal{M}_+(\Omega) \setminus \{0\}$ be an endemic equilibrium measure with mass $m = \mu(\Omega)$. Since $k(x, y) = Q(x)$, $K\mu(x) = mQ(x)$, and the stationary equation becomes

$$m(1 - m)Q(x)\,dx = \gamma(x)\mu(dx).$$

Hence, on $\Omega \setminus \{0\}$,

$$\mu(dx) = m(1 - m)\frac{Q(x)}{\gamma(x)}\,dx,$$

while the part supported on $\{0\}$ is not constrained by the stationary equation (Theorem 1.6). Therefore

$$\mu(dx) = m(1 - m)\frac{Q(x)}{\gamma(x)}\,dx + \beta_m\delta_0(dx), \qquad \beta_m \geq 0.$$

The mass constraint yields

$$\beta_m = m - \rho m(1 - m) = m(1 - \rho(1 - m)).$$

Thus $\beta_m \geq 0$ if and only if $m \geq 1 - \frac{1}{\rho}$. This proves both the expression of $m_\star$ and the claimed characterization of the endemic equilibrium measures. □

Remark 2.23. At $m = m_\star$, the singular part vanishes, so that

$$\mu_{m_\star}(dx) = m_\star(1 - m_\star)\frac{Q(x)}{\gamma(x)}\,dx,$$

whereas $\mu_1 = \delta_0$. Moreover,

$$\mu_m^{\mathrm{ac}}(\Omega) = \rho m(1-m), \qquad \mu_m^{\mathrm{sing}}(\Omega) = m - \rho m(1-m).$$

In particular, the singular mass increases strictly from 0 to 1 as m varies from $m_\star$ to 1.

Remark 2.24. The absolutely continuous mass

$$\mu_m^{\mathrm{ac}}(\Omega) = \rho m(1-m)$$

is maximal at $m = 1/2$. Hence, along the plateau $[m_\star, 1]$, it is non-monotone whenever $m_\star < 1/2$, equivalently $1 < \rho < 2$, and it is decreasing when $\rho \geq 2$. In contrast, the singular mass is always strictly increasing on $[m_\star, 1]$.

3 The evolution equation

We now turn to the study of the time evolution of the infected population. Throughout this section, we consider the strain-structured SIS model introduced above and focus on the associated nonlinear dynamics. The density of infected individuals $I(t,x)$ evolves according to the following integro-differential equation:

$$\tfrac{\partial}{\partial t} I(t,x) = S(t)\int_\Omega k(x,y)I(t,y)\,\mathrm{d}y - \gamma(x)I(t,x), \tag{3}$$

Proposition 3.1 (Local Existence and Uniqueness in L^1)**.** *Under hypotheses (K1) and (*Γ1*), for any initial profile $I_0 \in L^1(\Omega)$, there exists a maximal time $T_{max} > 0$ and a unique solution $I \in \mathcal{C}^1([0, T_{max}); L^1(\Omega))$ to equation* (3)*.*

Proof. We reformulate the problem as an abstract ordinary differential equation in the Banach space $E = L^1(\Omega)$:

$$\tfrac{dI}{dt} = F(I), \quad I(0) = I_0,$$

where the operator $F : L^1(\Omega) \to L^1(\Omega)$ is defined by:

$$F(f) = \left(1 - \int_\Omega f(z)dz\right) K(f) - \Gamma f.$$

1. F is locally Lipschitz. Let $f, g \in L^1(\Omega)$ and define $\psi(f) = \int_\Omega f(z)dz$. We estimate the difference in the L^1 norm:

$$F(f) - F(g) = (1 - \psi(f))K(f-g) + (\psi(g) - \psi(f))K(g) - \Gamma(f-g).$$

Using the boundedness of K and the boundedness of the multiplication by γ, we obtain:

$$\begin{aligned}\|F(f) - F(g)\|_{L^1} &\leq |1 - \psi(f)| \cdot \|K(f-g)\|_{L^1} + |\psi(g) - \psi(f)| \cdot \|K(g)\|_{L^1} + \|\Gamma(f-g)\|_{L^1} \\ &\leq \left(|1 - \psi(f)|\|K\| + \|K\|\|g\|_{L^1} + \overline{\gamma}\right)\|f - g\|_{L^1}.\end{aligned}$$

Consider a bounded ball $B_R = \{h \in L^1(\Omega) : \|h\|_{L^1} \leq R\}$. For any $f, g \in B_R$, the terms $|1 - \psi(f)|$ and $\|g\|_{L^1}$ are bounded. Therefore, there exists a constant C_R depending on R such that:

$$\|F(f) - F(g)\|_{L^1} \leq C_R \|f - g\|_{L^1}.$$

Consequently, the operator F is locally Lipschitz on $L^1(\Omega)$.

2. Conclusion. The Cauchy-Lipschitz theorem for Banach spaces guarantees the existence and uniqueness of a maximal solution $I \in \mathcal{C}^1([0, T_{max}); L^1(\Omega))$. □

The local well-posedness established in Proposition 3.1 ensures that solutions exist and are unique in a short time frame, a crucial first step for further dynamical studies. To extend this globally and analyze stability, we must first verify the non-negativity of $I(t, x)$, which aligns with the biological interpretation of densities as positive quantities.

Notation 3.2. We define:

$$t^* := \sup\{\tau \in [0, T_{max}) \mid \forall t \in [0, \tau], m(t) \leq 1\}.$$

Proposition 3.3 (Preservation of non-negativity)**.** *Let $I_0 \in L^1(\Omega)$ satisfy $I_0 \geq 0$ a.e. in Ω, with*

$$m(0) := \int_\Omega I_0(x)\, dx \in [0, 1],$$

and let $I \in \mathcal{C}^1([0, T_{\max}); L^1(\Omega))$ be the corresponding solution of (3). Then $t^* > 0$, and, for every $t \in [0, t^*)$,

$$I(t, x) \geq 0 \qquad \text{for a.e. } x \in \Omega.$$

Proof. We first check that $t^* > 0$. If $m(0) < 1$, this follows directly from the continuity of m. Assume now that $m(0) = 1$. Since $I_0 \geq 0$,

$$m'(0) = -\int_\Omega \gamma(x) I_0(x)\, dx \leq 0.$$

If the inequality is strict, then $m(t) < 1$ for all sufficiently small $t > 0$. If instead

$$\int_\Omega \gamma(x) I_0(x)\, dx = 0,$$

then $\gamma I_0 = 0$ a.e. Since $m(0) = 1$, the right-hand side of (3) vanishes at I_0, so $I(t) \equiv I_0$ is a solution. By uniqueness, it is the corresponding solution, and hence $m(t) = 1$. Thus $t^* > 0$ in all cases.

We now prove non-negativity on $[0, t^*)$, where $S(t) = 1 - m(t) \geq 0$. Recall the notation

$$x_- := \max\{-x, 0\}.$$

For $\varepsilon > 0$, consider the convex C^1-regularization

$$\Phi_\varepsilon(x) = \begin{cases} -x - \dfrac{\varepsilon}{2}, & x \leq -\varepsilon, \\ \dfrac{x^2}{2\varepsilon}, & -\varepsilon < x < 0, \\ 0, & x \geq 0. \end{cases}$$

Then $\Phi_\varepsilon \to x_-$ uniformly as $\varepsilon \to 0$, and

$$-1 \le \Phi_\varepsilon'(x) \le 0, \qquad \Phi_\varepsilon'(x) = 0 \quad \text{for } x \ge 0.$$

In particular, $0 \le x\Phi_\varepsilon'(x) \le x_-$.

Set

$$\mathcal{E}_\varepsilon(t) := \int_\Omega \Phi_\varepsilon(I(t,x))\,dx.$$

Differentiating along solutions yields

$$\frac{d}{dt}\mathcal{E}_\varepsilon(t) = \int_\Omega \Phi_\varepsilon'(I)\big[S(t)KI - \gamma I\big]\,dx.$$

Since $\gamma \ge 0$ and $I\Phi_\varepsilon'(I) \ge 0$, the recovery term is non-positive. Moreover, by Fubini,

$$\int_\Omega \Phi_\varepsilon'(I)(KI)\,dx = \int_\Omega I(y)\left(\int_\Omega k(x,y)\Phi_\varepsilon'(I(x))\,dx\right)dy.$$

Since $k \ge 0$ and $-1 \le \Phi_\varepsilon' \le 0$, the contribution of the set $\{I(y) \ge 0\}$ is non-positive, whereas on $\{I(y) < 0\}$,

$$I(y)\int_\Omega k(x,y)\Phi_\varepsilon'(I(x))\,dx \le I_-(y)\|k(\cdot,y)\|_{L^1}.$$

Therefore,

$$\int_\Omega \Phi_\varepsilon'(I)(KI)\,dx \le \overline{k}\int_\Omega I_-(y)\,dy, \qquad \overline{k} := \sup_{y\in\Omega}\|k(\cdot,y)\|_{L^1}.$$

Since $S(t) \ge 0$ on $[0,t^*)$,

$$\frac{d}{dt}\mathcal{E}_\varepsilon(t) \le \overline{k}S(t)\|I_-(t)\|_{L^1}.$$

Moreover, $I_0 \ge 0$ implies $\mathcal{E}_\varepsilon(0) = 0$. Integrating in time and letting $\varepsilon \to 0$ gives

$$\|I_-(t)\|_{L^1} \le \overline{k}\int_0^t S(\tau)\|I_-(\tau)\|_{L^1}\,d\tau.$$

Grönwall's lemma then yields

$$\|I_-(t)\|_{L^1} = 0, \qquad t \in [0,t^*),$$

which proves the claim. □

Proposition 3.4. *Assume that $m(0) \in [0,1)$ and $I_0 \in L^1_+(\Omega)$. Then $m \in \mathcal{C}^1([0,T_{\max}))$ and*

$$0 \le m(t) < 1, \qquad \textit{for all } t \in [0,T_{\max}).$$

Moreover, $T_{\max} = +\infty$.

Proof. Since $I \in \mathcal{C}^1([0,T_{\max}); L^1(\Omega))$, the map m is of class $\mathcal{C}^1$, with

$$m'(t) = (1-m(t))\int_\Omega (KI)(t,x)\,dx - \int_\Omega \gamma(x)I(t,x)\,dx.$$

By Proposition 3.3, $I(t,\cdot) \geq 0$ for $t < t^*$, and therefore $m(t) \geq 0$. Moreover, for $t < t^*$, since $1 - m(t) \geq 0$,

$$m'(t) \leq (1 - m(t)) \int_\Omega \int_\Omega k(x,y) I(t,y)\, dy\, dx \leq (1 - m(t))\, \overline{k}\, m(t).$$

Comparison with the logistic equation yields

$$m(t) \leq \frac{m(0) e^{\overline{k} t}}{1 - m(0) + m(0) e^{\overline{k} t}} < 1, \qquad \text{for} \qquad 0 \leq t < t^*.$$

In particular, m cannot reach 1 at any finite time. By the definition of t^* and the continuity of m, this implies $t^* = T_{\max}$. Hence $0 \leq m(t) < 1$ for all $t < T_{\max}$.

Since $I(t,\cdot) \geq 0$ on $[0, T_{\max})$, $\|I(t,\cdot)\|_{L^1(\Omega)} = m(t) < 1$. Thus the L^1-norm of the solution remains bounded on every bounded time interval. The blow-up alternative for the local solution therefore excludes $T_{\max} < +\infty$, and we conclude that $T_{\max} = +\infty$. □

Proposition 3.5 (Convergence to Disease-Free Equilibrium when $s(L_0) < 0$)**.** *In the case where $-\underline{\gamma} < s(L_0) < 0$, we have:*

$$m(t) \xrightarrow{t \to +\infty} 0 \quad \textit{and the convergence rate is exponential.}$$

Proof. Introduce the function $Q(t) = \langle \Phi, I(t) \rangle = \int_\Omega \Phi(x) I(t,x)\, dx$, where Φ is the positive eigenfunction of L_0^* associated with the eigenvalue $s(L_0)$ (given by Theorem 6.7). The derivative of $Q(t)$ is obtained as follows:

$$\tfrac{d}{dt} Q(t) = \langle \Phi, L_{m(t)} I(t) \rangle \leq \langle \Phi, L_0 I(t) \rangle = s(L_0) Q(t).$$

By direct application of Grönwall's lemma, we obtain: $Q(t) \leq Q(0) e^{s(L_0)t} \xrightarrow{t \to \infty} 0$. Thus, $Q(t) \to 0$, and since:

$$Q(t) \geq \underline{\Phi} m(t) \quad \text{where } \underline{\Phi} = \min\{\Phi(x) \mid x \in \Omega\} > 0,$$

we have : $m(t) \leq \frac{Q(0) e^{s(L_0)t}}{\underline{\Phi}}$ and the convergence rate is exponential. □

Proposition 3.6. *Suppose that $s(L_0) = -\underline{\gamma} < 0$. Then, for every initial condition $I_0 \in L^1_+(\Omega)$ with $m(0) \in [0,1)$, the corresponding solution $I(t)$ converges exponentially fast to zero.*

Proof. For $\theta \geq 1$, we have $L_{1-\theta} = \theta K - \Gamma$. Since $s(L_0) = -\underline{\gamma} < 0$ and the map $\theta \mapsto s(L_{1-\theta})$ is continuous, Proposition 2.3 ensures that there exists $\theta^* > 1$ such that $-\underline{\gamma} < s(L_{1-\theta^*}) < 0$. By Theorem 6.7, there exists a strictly positive dual eigenfunction $\Phi_{1-\theta^*} \in L^\infty_+(\Omega)$ such that

$$L^*_{1-\theta^*} \Phi_{1-\theta^*} = s(L_{1-\theta^*}) \Phi_{1-\theta^*}, \qquad \Phi_{1-\theta^*} \geq c > 0.$$

Define the functional $Q(t) := \langle \Phi_{1-\theta^*}, I(t) \rangle$. Since $m(t) \geq 0$ and $1 - \theta^* < 0$, we have the operator comparison $L_{m(t)} \leq K - \Gamma \leq L_{1-\theta^*}$. Differentiating $Q(t)$ and using the positivity of $\Phi_{1-\theta^*}$ gives

$$\tfrac{d}{dt} Q(t) = \langle \Phi_{1-\theta^*}, L_{m(t)} I(t) \rangle \leq \langle \Phi_{1-\theta^*}, L_{1-\theta^*} I(t) \rangle = s(L_{1-\theta^*}) Q(t).$$

An application of Grönwall's lemma yields $Q(t) \leq Q(0) e^{s(L_{1-\theta^*})t}$. Since $\Phi_{1-\theta^*}$ is bounded below by a positive constant, we have $\|I(t)\|_{L^1} \leq c^{-1} Q(t)$, which proves the exponential decay of $I(t)$. □

Proposition 3.7 (Algebraic decay in the critical case)**.** *Assume that $s(L_0) = 0$ and that $\underline{\gamma} > 0$. Then, for every non-negative initial condition $I_0 \in L^1(\Omega)$, with $m(0) \in [0,1]$, the solution satisfies*

$$m(t) = \|I(t)\|_{L^1(\Omega)} = O\left(\tfrac{1}{t}\right) \quad \text{as } t \to +\infty.$$

Proof. The zero solution being trivial, assume $I_0 \not\equiv 0$ and let $\Phi \in \mathcal{C}(\Omega)$ denote the strictly positive eigenfunction of the dual operator L_0^* associated with the eigenvalue 0, that is, $L_0^*\Phi = K^*\Phi - \Gamma\Phi = 0$. Recall the functional $Q(t) := \langle \Phi, I(t)\rangle = \int_\Omega \Phi(x) I(t,x)\,dx$. Differentiating $Q(t)$ with respect to time and using the main equation, we obtain:

$$\begin{aligned}
\tfrac{d}{dt}Q(t) &= \langle \Phi, (1-m(t))KI(t) - \Gamma I(t)\rangle \\
&= \langle \Phi, KI(t)\rangle - m(t)\langle \Phi, KI(t)\rangle - \langle \Phi, \Gamma I(t)\rangle \\
&= \underbrace{\langle K^*\Phi - \Gamma\Phi, I(t)\rangle}_{=0} - m(t)\langle \Phi, KI(t)\rangle \\
&= -m(t)\langle K^*\Phi, I(t)\rangle.
\end{aligned}$$

To estimate the term on the right-hand side, set $\Psi := K^*\Phi$. Since $L_0^*\Phi = 0$, we have $\Psi = K^*\Phi = \Gamma\Phi = \gamma\Phi$. Hence, using $\underline{\gamma} > 0$ and the strict positivity of Φ,

$$C_\Psi := \min_{x\in\Omega} \Psi(x) \geq \underline{\gamma} \min_{x\in\Omega} \Phi(x) > 0.$$

We can therefore bound the inner product as follows:

$$\langle K^*\Phi, I(t)\rangle = \int_\Omega \Psi(x) I(t,x)\,dx \geq C_\Psi \int_\Omega I(t,x)\,dx = C_\Psi\, m(t).$$

Substituting this into the derivative of $Q(t)$ yields the differential inequality:

$$\tfrac{d}{dt}Q(t) \leq -C_\Psi (m(t))^2. \tag{4}$$

Finally, we relate $m(t)$ back to $Q(t)$. Let $\overline{\Phi} := \max_{x\in\Omega} \Phi(x)$ and $\underline{\Phi} := \min_{x\in\Omega} \Phi(x)$. Then $Q(t) \leq \overline{\Phi}\, m(t)$, which implies $m(t) \geq \frac{Q(t)}{\overline{\Phi}}$. Injecting this into (4) gives:

$$\tfrac{d}{dt}Q(t) \leq -\tfrac{C_\Psi}{(\overline{\Phi})^2}(Q(t))^2.$$

Integrating this differential inequality yields $Q(t) \leq \frac{1}{\frac{C_\Psi}{(\overline{\Phi})^2}t + \frac{1}{Q(0)}}$. Finally, since $m(t) \leq \underline{\Phi}^{-1} Q(t)$, we conclude that

$$m(t) \leq \tfrac{1}{\underline{\Phi}} \frac{1}{\frac{C_\Psi}{(\overline{\Phi})^2}t + \frac{1}{Q(0)}} = O\left(\tfrac{1}{t}\right)$$

which completes the proof.

□

Remark 3.8. The previous argument relies on the uniform positivity of the recovery rate. When $\underline{\gamma} = 0$ and $s(L_0) = 0$, the long-time behavior is not determined by the present analysis and remains an open question.

Proposition 3.9 (Persistence of the epidemic when $s(L_0) > 0$)**.** *Assume that $s(L_0) > 0$. Then, for any non-trivial solution $I(t)$, the total infected mass satisfies*

$$\liminf_{t\to+\infty} m(t) > 0.$$

Proof. Let $\Phi \in C_+(\Omega)$, $\Phi \not\equiv 0$ be a positive eigenfunction of the dual operator L_0^* associated with the principal eigenvalue $s(L_0)$, and recall $Q(t) := \langle \Phi, I(t)\rangle$.

By positivity and continuity of Φ on the compact set Ω, there exist constants

$$0 < \underline{\Phi} \le \Phi(x) \le \overline{\Phi} < +\infty \quad \text{for all } x \in \Omega,$$

so that $\underline{\Phi}\, m(t) \le Q(t) \le \overline{\Phi}\, m(t)$.

Using the decomposition $L_{m(t)} = L_0 - m(t)K$ and the fact that $L_0^*\Phi = s(L_0)\Phi$, we compute

$$\tfrac{d}{dt}Q(t) = \langle \Phi, L_{m(t)}I(t)\rangle = s(L_0)Q(t) - m(t)\langle \Phi, KI(t)\rangle.$$

Let $\psi(y) = K^*\Phi(y)$. We have $0 \le \psi(y) \le \overline{\Phi}.\overline{k} < +\infty$. It follows that

$$\langle \Phi, KI(t)\rangle = \int_\Omega \psi(y) I(t,y)\, dy \le \overline{\Phi k}\, m(t).$$

Therefore,

$$\tfrac{d}{dt}Q(t) \ge s(L_0)Q(t) - \overline{\Phi}.\overline{k}\, m(t)^2.$$

We obtain the global differential inequality

$$\tfrac{d}{dt}Q(t) \ge s(L_0)Q(t) - \tfrac{\overline{\Phi}.\overline{k}}{\underline{\Phi}^2}\, Q(t)^2 \qquad \text{for all } t \ge 0. \tag{5}$$

$Q(t)$ is actually a super solution of the logistic equation. Consequently, by Proposition 6.3 one has:

$$Q(t) \ge \frac{C}{1+\left(\frac{C-Q(0)}{Q(0)}\right)e^{-s(L_0)t}} \qquad \text{with} \qquad C := \tfrac{\underline{\Phi}^2 s(L_0)}{\overline{\Phi}.\overline{k}}$$

Which implies:

$$\liminf_{t\to+\infty} Q(t) \ge \lim_{t\to+\infty} \frac{C}{1+\left(\frac{C-Q(0)}{Q(0)}\right)e^{-s(L_0)t}} = C > 0.$$

Using $Q(t) \le \overline{\Phi} m(t)$, we conclude that $\liminf_{t\to+\infty} m(t) \ge \frac{s(L_0)\,\underline{\Phi}^2}{\overline{k}.\overline{\Phi}^2} > 0$, which proves the persistence of the epidemic. □

Proposition 3.10. *Assume that $\underline{\gamma} > 0$ and that $s(L_0) > 0$. Then there exists a unique $m^* \in (0,1)$ such that $s(L_{m^*}) = 0$, and for any nontrivial solution $I(t)$ (that is, $m(0) \in (0,1)$), one has*

$$\liminf_{t\to\infty} |m(t) - m^*| = 0.$$

Proof. Since $\underline{\gamma} > 0$ and $s(L_{m^*}) = 0$, Proposition 2.8 ensures that there exists $\delta > 0$ such that $m \mapsto s(L_m)$ is strictly decreasing on $(m^* - \delta, m^* + \delta)$, and for each m in this interval, the adjoint operator L_m^* admits a continuous strictly positive eigenfunction Φ_m.

Assume by contradiction that

$$\liminf_{t\to\infty} |m(t) - m^*| > 0.$$

Then there exist $\eta \in (0, \delta)$ and $T \geq 0$ such that either

$$m(t) \leq m_- := m^* - \eta \quad \forall t \geq T, \qquad \text{or} \qquad m(t) \geq m_+ := m^* + \eta \quad \forall t \geq T.$$

Let $m_\pm$ denote either of these two values. In both cases, writing $L_{m(t)} = L_{m_\pm} + (m_\pm - m(t))K$, and defining $H_\pm(t) := \langle \Phi_{m_\pm}, I(t) \rangle$, we obtain

$$\tfrac{d}{dt} H_\pm(t) = s(L_{m_\pm}) H_\pm(t) + (m_\pm - m(t)) \langle \Phi_{m_\pm}, KI(t) \rangle.$$

Since $K \geq 0$, $I(t) \geq 0$ and $\Phi_{m_\pm} > 0$, we have $\langle \Phi_{m_\pm}, KI(t) \rangle \geq 0$, hence

$$\tfrac{d}{dt} H_-(t) \geq s(L_{m_-}) H_-(t), \qquad \tfrac{d}{dt} H_+(t) \leq s(L_{m_+}) H_+(t).$$

Because $s(L_{m_-}) > 0$ and $H_-(T) > 0$, we deduce that $H_-(t) \to +\infty$, while $H_-(t) \leq \|\Phi_{m_-}\|_{L^\infty}$, a contradiction. Similarly, since $s(L_{m_+}) < 0$, we obtain $H_+(t) \to 0$, while

$$H_+(t) \geq \underline{\Phi_{m_+}} m(t) \geq \underline{\Phi_{m_+}} m_+ > 0,$$

again a contradiction.

Therefore, for every $\eta \in (0, \delta)$ and every $T \geq 0$, there exists $t \geq T$ such that $|m(t) - m^*| < \eta$, which yields

$$\liminf_{t \to \infty} |m(t) - m^*| = 0.$$

□

Lyapunov functionals of Volterra type and positive operator methods have been used to establish global stability of endemic equilibria in infinite-dimensional epidemic models; see Thieme [38]. In our setting, the following proposition establishes convergence to the endemic equilibrium under an additional balanced transmission condition.

Proposition 3.11 (Convergence under a balanced transmission condition)**.** *Assume that* $\underline{\gamma} > 0$*,* $R_0 > 1$*, and that*

$$\int_\Omega \frac{k(x,y)}{\gamma(x)}\, dx = R_0, \qquad y \in \Omega.$$

Let I^* *denote the unique endemic equilibrium and set*

$$m^* := \int_\Omega I^*(x)\, dx = 1 - \frac{1}{R_0}.$$

Let $I_0 \in L^1_+(\Omega) \setminus \{0\}$ *satisfy* $0 < m(0) < 1$ *and* $\mathcal{V}(I_0) < +\infty$*, where*

$$\mathcal{V}(I) := \int_\Omega \frac{I^*(x)}{\gamma(x)} h\left(\frac{I(x)}{I^*(x)}\right) dx, \qquad h(z) := z - 1 - \log z.$$

Then the corresponding solution satisfies $m(t) \xrightarrow{t \to +\infty} m^*$*. Moreover,* $I(t, \cdot) \xrightarrow[L^1(\Omega)]{t \to +\infty} I^*$*.*

Proof. Since $\int_\Omega \frac{k(x,y)}{\gamma(x)}\, dx = R_0$, the function $\Phi(x) := \frac{1}{\gamma(x)}$ satisfies $K^*\Phi = R_0$. As $1 - m^* = \frac{1}{R_0}$, we obtain

$$L^*_{m^*}\Phi = (1 - m^*) K^* \Phi - \Gamma \Phi = 0.$$

Moreover, the endemic equilibrium satisfies $\frac{1}{R_0} K I^* = \Gamma I^*$. Define $q(x,y) := \frac{1}{R_0} \frac{k(x,y)}{\gamma(x)} I^*(y)$. The assumption on k gives $\int_\Omega q(x,y)\,dx = I^*(y)$, while the equilibrium equation gives $\int_\Omega q(x,y)\,dy = I^*(x)$. Hence

$$\int_{\Omega^2} q(x,y)\,dx\,dy = m^*.$$

Set $z(t,x) := \frac{I(t,x)}{I^*(x)}$ and $a(t) := R_0(1-m(t))$. Differentiating $\mathcal{V}$ along the solution, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(I(t)) &= \int_\Omega \frac{1}{\gamma(x)}\left(1 - \frac{1}{z(t,x)}\right)\partial_t I(t,x)\,dx \\ &= a(t)\int_{\Omega^2} q(x,y) z(t,y)\left(1 - \frac{1}{z(t,x)}\right)\,dx\,dy \\ &\quad - \int_\Omega I^*(x)(z(t,x)-1)\,dx.\end{aligned}$$

Using the marginal identities for q, this becomes

$$\frac{d}{dt}\mathcal{V}(I(t)) = a(t)\big(m(t) - D(t)\big) - \big(m(t) - m^*\big),$$

where

$$D(t) := \int_{\Omega^2} q(x,y)\frac{z(t,y)}{z(t,x)}\,dx\,dy.$$

Since q/m^* is a probability measure on Ω^2, Jensen's inequality gives

$$\begin{aligned}\frac{D(t)}{m^*} &= \int_{\Omega^2} \frac{q(x,y)}{m^*} \exp\left(\log z(t,y) - \log z(t,x)\right)\,dx\,dy \\ &\geq \exp\left(\frac{1}{m^*}\int_{\Omega^2} q(x,y)\big(\log z(t,y) - \log z(t,x)\big)\,dx\,dy\right).\end{aligned}$$

The two marginals of q are both equal to I^*, hence the exponent vanishes. Therefore $D(t) \geq m^*$. Consequently,

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(I(t)) &\leq a(t)\big(m(t)-m^*\big) - \big(m(t)-m^*\big) \\ &= \big(R_0(1-m(t)) - 1\big)\big(m(t)-m^*\big) \\ &= -R_0\big(m(t)-m^*\big)^2.\end{aligned}$$

Since $\mathcal{V} \geq 0$, it follows that

$$\int_0^{+\infty} \big(m(t)-m^*\big)^2\,dt \leq \frac{\mathcal{V}(I_0)}{R_0} < +\infty.$$

On the other hand,

$$m'(t) = (1-m(t))\int_\Omega K I(t,x)\,dx - \int_\Omega \gamma(x) I(t,x)\,dx,$$

and, since $0 \leq m(t) \leq 1$, $|m'(t)| \leq \overline{k} + \overline{\gamma} < +\infty$. Thus m is uniformly continuous on $\mathbb{R}_+$. Since $m - m^* \in L^2(\mathbb{R}_+)$, Barbalat's lemma yields

$$m(t) \longrightarrow m^*.$$

It remains to prove the convergence of the whole profile. Since $\underline{\gamma} > 0$, one has $s(L_{m^*}) = 0 > -\underline{\gamma}$. Moreover, since K is compact, 0 is isolated from the essential spectrum. Proposition 2.7 shows that

$$\ker L_{m^*} = \operatorname{span}\{I^*\}.$$

Let $\Phi > 0$ satisfy $L_{m^*}^*\Phi = 0$, normalized so that $\langle \Phi, I^* \rangle = 1$. If there existed a generalized eigenvector u such that $L_{m^*} u = I^*$, then

$$1 = \langle \Phi, I^* \rangle = \langle L_{m^*}^* \Phi, u \rangle = 0,$$

which is impossible. Hence 0 is algebraically simple.

It remains to exclude non-zero spectral values on the imaginary axis. Assume that

$$L_{m^*} f = i\tau f, \qquad \tau \neq 0.$$

Then $(\gamma + i\tau) f = (1 - m^*) K f$, and therefore

$$\sqrt{\gamma^2 + \tau^2}\, |f| \leq (1 - m^*) K |f|.$$

Pairing with Φ and using $(1 - m^*) K^* \Phi = \gamma \Phi$, we obtain

$$\int_\Omega \sqrt{\gamma(x)^2 + \tau^2}\, \Phi(x) |f(x)|\, dx \leq \int_\Omega \gamma(x) \Phi(x) |f(x)|\, dx,$$

which is impossible since $\tau \neq 0$, $\Phi > 0$, and $f \not\equiv 0$. Thus 0 is the only spectral value of L_{m^*} on the imaginary axis.

Since L_{m^*} is bounded and the remainder of its spectrum is contained in $\{z \in \mathbb{C} : \operatorname{Re} z \leq -\omega\}$ for some $\omega > 0$, the spectral mapping theorem yields

$$\| e^{t L_{m^*}} (I - P) \|_{\mathcal{L}(L^1)} \leq C e^{-\omega t}.$$

The projection P has rank one and satisfies $\operatorname{Ran}(P) = \operatorname{span}\{I^*\}$.

Since $L_{m(t)} = L_{m^*} + (m^* - m(t)) K$, Duhamel's formula gives

$$I(t) = e^{t L_{m^*}} I_0 + \int_0^t e^{(t-s) L_{m^*}} (m^* - m(s)) K I(s)\, ds.$$

Applying $I - P$, we obtain

$$\begin{aligned} \| (I - P) I(t) \|_{L^1} \leq\ & C e^{-\omega t} \| (I - P) I_0 \|_{L^1} \\ & + C \|K\| \int_0^t e^{-\omega (t-s)} |m(s) - m^*| \| I(s) \|_{L^1}\, ds. \end{aligned}$$

Since $\|I(s)\|_{L^1} = m(s) \leq 1$ and $m(s) \longrightarrow m^*$, the convolution term tends to zero. Therefore $(I - P) I(t) \longrightarrow 0$ in $L^1(\Omega)$.

Writing $P I(t) = \alpha(t) I^*$, we have

$$m(t) = \alpha(t) m^* + \int_\Omega (I - P) I(t, x)\, dx.$$

Since $m(t) \to m^*$ and $(I - P) I(t) \to 0$ in $L^1(\Omega)$, it follows that $\alpha(t) \longrightarrow 1$. Consequently,

$$I(t, \cdot) \longrightarrow I^* \qquad \text{in } L^1(\Omega).$$

□

A further convergence criterion is obtained when the total transmission rate is proportional to the recovery rate. The convergence of the total infected mass follows from the bounds on individual carrying capacities established in [1, Theorem 3.2].

Proposition 3.12 (Convergence for a constant ratio β/γ)**.** *Under the standing assumptions on k and γ, assume that $\underline{\gamma} > 0$ and that, for some $C > 1$,*

$$\beta(y) := \int_\Omega k(x,y)\,dx = C\gamma(y), \qquad y \in \Omega.$$

Then $\mathcal{R}_0 = C$. For every $I_0 \in L^1_+(\Omega) \setminus \{0\}$ with $0 < m(0) < 1$, the corresponding solution satisfies

$$m(t) \longrightarrow m^* := 1 - \frac{1}{C}, \qquad \|I(t,\cdot) - I^*\|_{L^1(\Omega)} \longrightarrow 0 \qquad \text{as } t \to +\infty,$$

where I^* is the unique endemic equilibrium.

Proof. Since $N^*\mathbf{1} = C\mathbf{1}$, the Collatz–Wielandt inequalities yield $\mathcal{R}_0 = r(N) = C$.

Integrating the evolution equation gives

$$m'(t) = \big(C(1-m(t)) - 1\big)\int_\Omega \gamma(x) I(t,x)\,dx = C\big(m^* - m(t)\big)\int_\Omega \gamma(x) I(t,x)\,dx.$$

Thus m is monotone and remains between $m(0)$ and m^*. Moreover, $\int_\Omega \gamma(x)I(t,x)\,dx \geq \underline{\gamma}\min\{m(0), m^*\} > 0$, which implies $m(t) \to m^*$.

Finally, the spectral-gap argument in the last part of the proof of Proposition 3.11 only requires uniformly positive recovery and convergence of the mass to m^*. It therefore applies here and gives $I(t,\cdot) \to I^*$ in $L^1(\Omega)$. □

Remark 3.13. The condition $\beta/\gamma = C$ involves the recovery rate $\gamma(y)$ of the infecting strain, while the balanced transmission condition involves $\gamma(x)$, the recovery rate of the newly infected strain. These conditions do not imply one another in general and coincide when γ is constant. The present proposition also requires no finite initial entropy assumption.

We now show that such an additional structure is genuinely needed: under the general assumptions of the model, the unique endemic equilibrium may be unstable.

Proposition 3.14 (An unstable endemic equilibrium)**.** *There exists a regular supercritical model satisfying* (K1)–(K3) *and* (Γ1) *whose unique endemic equilibrium is unstable. In particular, global convergence towards the endemic equilibrium fails, and there exist solutions for which the total infected mass does not converge to m^*.*

Proof. Let

$$\Omega = \bigcup_{i=1}^{4} \Omega_i, \qquad \Omega_i = [2i-2, 2i-1],$$

and, for $x \in \Omega_i$ and $y \in \Omega_j$, set $k(x,y) = a_{ij}$ and $\gamma(x) = \gamma_i$, where

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (20, 15, 0.2, 0.2)$$

and

$$A = (a_{ij}) = \begin{pmatrix} 10 & \varepsilon & \varepsilon & 0.1 \\ 0.1 & 20 & \varepsilon & \varepsilon \\ \varepsilon & 0.5 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 5 & \varepsilon \end{pmatrix}, \qquad \varepsilon = 10^{-3}.$$

Since the connected components Ω_i are disjoint, γ is continuous on Ω, with $\underline{\gamma} = 0.2$, while k is strictly positive and the associated operator K has finite rank. Hence the assumptions of the regular framework are satisfied.

For

$$M_i(t) := \int_{\Omega_i} I(t,x)\,dx, \qquad \mathbf{M} = (M_1, \dots, M_4)^{\mathsf{T}},$$

the dynamics reduces exactly to

$$\dot{\mathbf{M}} = (1 - \mathbf{1}^{\mathsf{T}}\mathbf{M})A\mathbf{M} - \Gamma\mathbf{M}, \qquad \Gamma = \mathrm{diag}(20, 15, 0.2, 0.2).$$

Moreover,

$$R_0 = r(\Gamma^{-1}A) \simeq 1.33750 > 1,$$

so that the continuous model admits a unique endemic equilibrium. Its total mass is

$$m^* = 1 - \frac{1}{R_0} \simeq 0.25234.$$

The Jacobian of the reduced system at the corresponding equilibrium $\mathbf{M}^*$ is

$$J^* = (1 - m^*)A - \Gamma - (A\mathbf{M}^*)\mathbf{1}^{\mathsf{T}}.$$

A validated computation of its characteristic polynomial gives

$$p(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4,$$

with

$$a_1 \in [13.1943, 13.1945], \qquad a_2 \in [8.1406, 8.1409],$$
$$a_3 \in [2.4357, 2.4359], \qquad a_4 \in [2.3584, 2.3586].$$

The first column of the associated Routh table has signs

$$+, \qquad +, \qquad +, \qquad -, \qquad +.$$

Hence, by the Routh–Hurwitz criterion, J^* has exactly two eigenvalues in the open right half-plane and two in the open left half-plane. Thus $\mathbf{M}^*$ is a hyperbolic saddle. By the stable manifold theorem, there exist positive initial conditions arbitrarily close to $\mathbf{M}^*$ whose trajectories do not converge to $\mathbf{M}^*$.

Since the finite-dimensional system above is an exact invariant reduction of the continuous model, the corresponding endemic equilibrium of the continuous model is not globally attractive.

Moreover, the last part of the proof of Proposition 3.11 shows that, in the regular case, the convergence

$$m(t) \longrightarrow m^*$$

implies

$$I(t,\cdot) \longrightarrow I^* \qquad \text{in } L^1(\Omega),$$

independently of the balanced transmission condition. Therefore, any trajectory which does not converge to the endemic equilibrium cannot satisfy $m(t) \to m^*$. Thus the recurrence result

$$\liminf_{t\to+\infty} |m(t) - m^*| = 0$$

obtained in Proposition 3.10 cannot, in general, be strengthened to convergence of $m(t)$.

□

4 Numerical simulations

The purpose of this section is to illustrate the long-time behavior obtained in the previous sections and to explore numerically the convergence properties that remain open in the general case. We deliberately restrict ourselves to a small number of examples.

4.1 Numerical scheme

We take $\Omega = [0,1]$ and discretize the trait variable on a uniform grid $x_1, \dots, x_N$. The integral operator is approximated by a quadrature rule, leading to the finite-dimensional system

$$\dot{\mathbf{I}} = \left(1 - \mathbf{w}^\top \mathbf{I}\right) K\mathbf{I} - \Gamma\mathbf{I},$$

where

$$K_{ij} = k(x_i, x_j) w_j, \qquad \Gamma_{ij} = \gamma(x_i)\delta_{ij},$$

and $\mathbf{w}$ denotes the vector of quadrature weights. We use the trapezoidal rule in space and the adaptive `RK45` method from `scipy.integrate.solve_ivp` for the time integration. The spectral bound $s(L_m)$ is approximated by the largest real part of the eigenvalues of the matrix $(1-m)K - \Gamma$.

4.2 Long-time behavior

We first consider an example for which the convergence results of Section 3 apply. Let

$$\gamma(x) = 1 + x$$

and

$$k_{\mathcal{R}}(x,y) = \mathcal{R}\,\gamma(x)\left(1 + \frac{1}{2}\cos(2\pi x)\cos(2\pi y)\right).$$

This kernel genuinely depends on both variables, but satisfies

$$\int_0^1 \frac{k_{\mathcal{R}}(x,y)}{\gamma(x)}\,dx = \mathcal{R} \qquad \text{for every } y \in [0,1].$$

Consequently, the parameter $\mathcal{R}$ coincides with the basic reproduction number of the model.

We consider successively $\mathcal{R} < 1$, $\mathcal{R} = 1$, and $\mathcal{R} > 1$. In the subcritical regime, the numerical solution exhibits the exponential extinction predicted by the theory. At the critical value $\mathcal{R} = 1$, the decay is substantially slower and is consistent with the estimate $m(t) = O(t^{-1})$. Finally, when $\mathcal{R} > 1$, the total infected mass converges to

$$m^* = 1 - \frac{1}{\mathcal{R}},$$

while the whole distribution $I(t,\cdot)$ converges towards the corresponding endemic equilibrium. These three behaviors are illustrated in Figure 6.

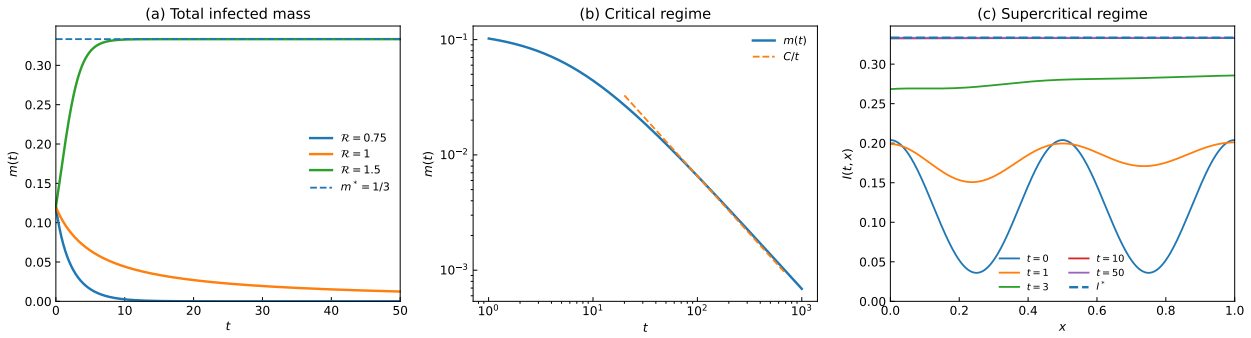


Figure 6: Long-time dynamics when the infectious potential is constant. The subcritical, critical and supercritical regimes illustrate, respectively, exponential extinction, the slower critical decay, and convergence towards the endemic equilibrium.

We finally illustrate the unstable example constructed in Proposition 3.14. Since the dynamics of the four masses is an exact reduction of the continuous model, we integrate

$$\dot{\mathbf{M}} = (1 - \mathbf{1}^{\mathsf{T}}\mathbf{M})A\mathbf{M} - \Gamma\mathbf{M}$$

from the initial condition

$$\mathbf{M}(0) = (0.05, 0.05, 0.05, 0.05)^{\mathsf{T}}.$$

The resulting trajectory is shown in Figure 7. After a transient, the total infected mass exhibits persistent numerical oscillations around the endemic mass $m^* \simeq 0.25234$, approximately between 0.20410 and 0.32025, with period 14.32.

These simulations are not used to establish the instability of the endemic equilibrium, which follows from Proposition 3.14. They rather illustrate the type of nonconvergent dynamics that may occur once global attractivity is lost. In particular, the observed oscillations remain consistent with

$$\liminf_{t \to +\infty} |m(t) - m^*| = 0.$$

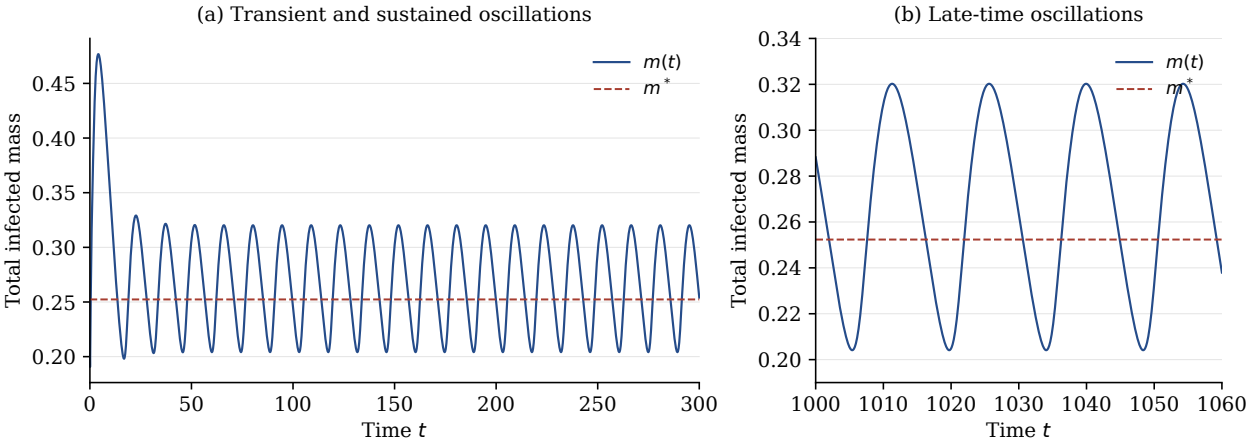


Figure 7: Numerical dynamics for the unstable example of Proposition 3.14. Starting from equal masses $M_i(0) = 0.05$, the total infected mass displays sustained numerical oscillations around the endemic mass m^*. The right panel shows a late-time window.

4.3 Concentration in the singular limit

We finally illustrate how measure-valued equilibria may arise when approaching the singular regime. To keep the mechanism transparent, we consider the kernel

$$k(x, y) = 1$$

and a family of regularized recovery rates

$$\gamma_\varepsilon(x) = \gamma_0(x) + \varepsilon, \qquad \varepsilon > 0.$$

For each $\varepsilon > 0$, the problem is regular and the endemic equilibrium, whenever $\mathcal{R}_{0,\varepsilon} > 1$, is explicitly given by

$$m_\varepsilon^* = 1 - \frac{1}{\mathcal{R}_{0,\varepsilon}}, \qquad I_\varepsilon^*(x) = \frac{m_\varepsilon^*}{\mathcal{R}_{0,\varepsilon}\gamma_\varepsilon(x)},$$

where

$$\mathcal{R}_{0,\varepsilon} = \int_0^1 \frac{dx}{\gamma_\varepsilon(x)}.$$

In order to isolate the concentration in trait space phenomenon, we plot the normalized equilibrium profiles

$$p_\varepsilon(x) := \frac{I_\varepsilon^*(x)}{m_\varepsilon^*} = \frac{1}{\mathcal{R}_{0,\varepsilon}\gamma_\varepsilon(x)},$$

which all have unit mass.

We consider two examples. First,

$$\gamma_0(x) = \frac{1 - \cos(2\pi x)}{2},$$

for which the zero-recovery set is $Z = \{0, 1\}$. As ε decreases, the equilibrium mass becomes increasingly concentrated near the two points of Z. Second, for

$$\gamma_0(x) = \frac{1 + \sin(2\pi x)}{2},$$

one has $Z = \{3/4\}$, and the normalized equilibrium develops a single sharp peak around this point. These computations do not establish a general singular-limit result, but they illustrate the emergence of concentrated profiles and are consistent with the measure-valued description of endemic equilibria obtained in Section 2.3.

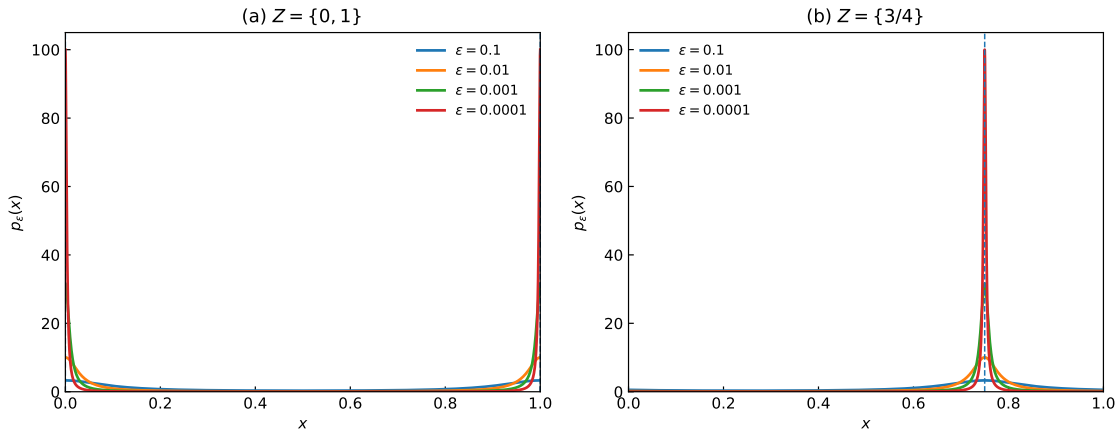


Figure 8: Concentration of the normalized endemic profiles $p_\varepsilon = I_\varepsilon^*/m_\varepsilon^*$ as the recovery rate approaches the singular regime. Left: $Z = \{0, 1\}$. Right: $Z = \{3/4\}$.

5 Conclusion

In this paper, we have studied the spectral structure and long-time dynamics of a strain-structured SIS model in which transmission and mutation are described by a nonlocal kernel. The family of operators $L_m = (1-m)K - \Gamma$ provides a common framework for the epidemic threshold and the characterization of endemic states.

The spectral bound $s(L_0)$ determines the main dynamical regimes. In the regular case, its sign is characterized by the classical next-generation operator. In the singular case $\underline{\gamma} = 0$, we introduced the generalized quantity

$$\mathcal{R}_0 = \lim_{\lambda \downarrow 0} r\Big(K(\lambda I + \Gamma)^{-1}\Big),$$

and proved that $s(L_0) > 0$ if and only if $\mathcal{R}_0 > 1$, whereas $\mathcal{R}_0 \leq 1$ corresponds to $s(L_0) = 0$.

At the nonlinear level, we established exponential extinction when $s(L_0) < 0$, the estimate $m(t) = O(t^{-1})$ in the regular critical case, and uniform persistence when $s(L_0) > 0$. In the regular supercritical regime, there is a unique endemic equilibrium, and the total infected mass satisfies $\liminf_{t\to+\infty} |m(t) - m^*| = 0$. Under the additional balanced transmission condition

$$\int_\Omega \frac{k(x,y)}{\gamma(x)}\, dx = \mathcal{R}_0, \qquad y \in \Omega,$$

we proved convergence of both the total mass and the infected profile:

$$m(t) \longrightarrow m^*, \qquad I(t,\cdot) \longrightarrow I^* \quad \text{in } L^1(\Omega).$$

We also constructed a regular supercritical example satisfying the general assumptions in which the unique endemic equilibrium is unstable. This rules out a general convergence theorem for the infected mass and profile under these assumptions alone. Numerical simulations of this example display sustained oscillations of the total infected mass. Thus uniqueness of the endemic equilibrium does not imply global attractivity, and the recurrence property of $m(t)$ is compatible with nonconvergent dynamics.

In the singular case, endemic equilibria are naturally described as positive Radon measures. We showed that their singular components must be supported on the zero-recovery set $Z = \{x \in \Omega : \gamma(x) = 0\}$, and derived criteria determining whether an equilibrium may or must charge this set. The possible appearance of a critical spectral plateau and singular endemic states distinguishes this regime from uniformly positive recovery.

Further questions therefore concern broader structural conditions ensuring convergence, as well as the existence, stability, and bifurcations of periodic solutions. In the singular regime, important directions include the critical dynamics and the selection of endemic measures, together with their relation to the spectral plateau and concentration on Z. Extending the analysis to non-compact trait spaces would also require addressing the possible loss of compactness and escape of mass.

6 Appendix

This appendix gathers several technical results and complementary arguments used throughout the paper. They are included here in order to keep the main text focused on the principal ideas and results.

Definition 6.1 (Function of type K)**.** Let $f : \mathbb{R} \times D \to \mathbb{R}^n$, where D is an open subset of $\mathbb{R}^n$, and write $f = (f_1, \ldots, f_n)$. The function f is said to be of type K in D if, for each index i, for all $t \in \mathbb{R}$ and all $a, b \in D$ such that

$$a \leq b \quad \text{and} \quad a_i = b_i,$$

one has

$$f_i(t, a) \leq f_i(t, b).$$

Remark 6.2. In the one-dimensional case ($n = 1$), this condition is automatically satisfied, since $a \leq b$ and $a_1 = b_1$ imply $a = b$.

This notion allows comparing solutions of the differential system

$$x' = f(t, x)$$

with sub- and supersolutions satisfying the corresponding differential inequalities.

The following useful result is taken from [36, Theorem B.1, p261]

Proposition 6.3. *Let $f : \mathbb{R} \times D \to \mathbb{R}^n$ be continuous and of type K, where $D \subset \mathbb{R}^n$ is open. Assume that the initial value problems associated with $x' = f(t, x)$ have unique local solutions. Let $x(\cdot)$ be a solution of*

$$x' = f(t, x)$$

defined on $[a, b]$.

If $z : [a, b] \to \mathbb{R}^n$ is continuous on $[a, b]$, differentiable on (a, b), satisfies

$$z'(t) \leq f(t, z(t)) \quad \text{on } (a, b),$$

and $z(a) \leq x(a)$, then $z(t) \leq x(t)$ for all $t \in [a, b]$.

The analogous statement holds for the reverse inequality $\geq$.

We also need spectral analysis results about positive operator. First we recall the following standard result, see [[35], Chap 6, Theorem 1.10 and Corollary 1.11].

Proposition 6.4. *Let E be a Banach lattice and let A be the generator of a positive strongly continuous semigroup $(T(t))_{t\geq 0}$. Let $B \in \mathcal{L}(E)$ be a positive operator ($B \geq 0$).*

Then the following assertions hold:

(i) (Spectral Bound Property) *If $s(A) > -\infty$, then the spectral bound belongs to the spectrum, i.e., $s(A) \in \sigma(A)$.*

(ii) (Domination of Semigroups) *The operator $A + B$ generates a positive strongly continuous semigroup $(S(t))_{t\geq 0}$ such that*

$$0 \leq T(t) \leq S(t) \qquad \text{for all } t \geq 0.$$

(iii) (Monotonicity of Spectral Bound) *The spectral bound increases under positive perturbation:*

$$s(A) \leq s(A + B).$$

Moreover, for all $\lambda > s(A + B)$, the resolvents satisfy the inequality:

$$0 \leq R(\lambda, A) \leq R(\lambda, A + B).$$

Proposition 6.5. *Let (X, Σ, μ) be a σ-finite measure space and let $E := L^p(\mu)$, with $1 \le p < +\infty$. Let $T \in \mathcal{L}(E)$ be a positive kernel operator of the form*

$$Tf(x) = \int_X K(x,y) f(y)\, d\mu(y),$$

where $K \ge 0$ is $(\Sigma \times \Sigma)$-measurable.

Recall that T is said to be irreducible *if the only closed ideals of $L^p(\mu)$ invariant under T are $\{0\}$ and $L^p(\mu)$. Equivalently, there is no measurable set $S \in \Sigma$ with*

$$\mu(S) > 0, \qquad \mu(X \setminus S) > 0,$$

such that

$$T\big(L^p(S)\big) \subset L^p(S).$$

Then T is irreducible if and only if

$$\int_{X\setminus S} \left(\int_S K(x,y)\, d\mu(y) \right) d\mu(x) > 0$$

for every measurable set $S \in \Sigma$ satisfying

$$\mu(S) > 0, \qquad \mu(X \setminus S) > 0.$$

This characterization is classical; see, for instance, Schaefer [35, Chapter V, Example 1.4].

We review a Krein-Rutman theorem from [35, Chapter V - Theorem 6.6]. We present hereafter two versions of the Krein-Rutman theorem, specifically adapted to the framework of operators generating positive semigroups in L^1. For modern expositions and extensions of these spectral properties in the context of population dynamics, we refer to [22].

Theorem 6.6. *Let $E := L^p(\mu)$, where $1 \le p \le +\infty$, and (X, Σ, μ) is a σ-finite measure space. Suppose $T \in \mathcal{L}(E)$ is an operator given by a $(\Sigma \times \Sigma)$-measurable kernel $K \ge 0$, satisfying the following two assumptions: T is power compact and irreducible.*

Then $r(T) > 0$ is an eigenvalue of T with a unique normalized eigenfunction f satisfying $f(s) > 0$ a.e. (μ). Moreover, if $K(s,t) > 0$ a.e. $(\mu \otimes \mu)$, then every other eigenvalue λ of T satisfies $|\lambda| < r(T)$.

Theorem 6.7 (Krein–Rutman in L^1)**.** *Let $\mathcal{L} = \mathcal{K} - \mathcal{G}$ on $L^1(X)$, where $\mathcal{K}$ is positive, compact and irreducible, $-\mathcal{G}$ bounded and resolvent positive with $s(-\mathcal{G}) < s(\mathcal{L}) < +\infty$. Then $s(\mathcal{L})$ is an eigenvalue of $\mathcal{L}$. There exist*

$$I \in L^1_+(X),\ I > 0 \text{ a.e.}, \qquad \Phi \in L^\infty_+(X),$$

such that

$$\mathcal{L}I = s(\mathcal{L})I, \qquad \mathcal{L}^*\Phi = s(\mathcal{L})\Phi,$$

with normalization $\langle \Phi, I \rangle = 1$.

Proof. By positivity of the semigroup, $s(\mathcal{L}) \in \sigma(\mathcal{L})$. Let $\lambda_n \downarrow s(\mathcal{L})$ and choose $g_n \ge 0$, $\|g_n\|_{L^1} \le 1$, such that

$$u_n := \tfrac{R(\lambda_n, \mathcal{L}) g_n}{\|R(\lambda_n, \mathcal{L}) g_n\|_{L^1}}$$

satisfies $\|u_n\|_{L^1} = 1$ and $u_n \geq 0$.

From the resolvent equation,

$$u_n = \mathcal{K}_{\lambda_n} u_n + \eta_n, \qquad \mathcal{K}_{\lambda_n} := (\lambda_n I + \mathcal{G})^{-1}\mathcal{K},$$

with $\eta_n \to 0$ in L^1. Since $\mathcal{K}_{\lambda_n} \to \mathcal{K}_s$ in operator norm and (u_n) is bounded in L^1,

$$\|\mathcal{K}_{\lambda_n} u_n - \mathcal{K}_s u_n\|_{L^1} \leq \|\mathcal{K}_{\lambda_n} - \mathcal{K}_s\|_{\mathcal{L}(L^1)} \|u_n\|_{L^1} \longrightarrow 0.$$

As $\mathcal{K}_s$ is compact, $(\mathcal{K}_s u_n)$ is relatively compact in L^1; hence so is $(\mathcal{K}_{\lambda_n} u_n)$. Hence (u_n) is also relatively compact, and up to extraction,

$$u_n \to h_0 \quad \text{in } L^1,$$

with $h_0 \geq 0$ and $\|h_0\|_{L^1} = 1$.

Since $\lambda \mapsto (\lambda I + \mathcal{G})^{-1}$ is continuous in operator norm, we have

$$\mathcal{K}_{\lambda_n} \to \mathcal{K}_{s(\mathcal{L})} \quad \text{in } \mathcal{L}(L^1(\Omega)).$$

Passing to the limit in

$$u_n = \mathcal{K}_{\lambda_n} u_n + \eta_n,$$

we obtain

$$h_0 = \mathcal{K}_{s(\mathcal{L})} h_0,$$

hence $\mathcal{L}h_0 = s(\mathcal{L})h_0$. By irreducibility, $h_0 > 0$ a.e.

The dual eigenfunction is obtained similarly by weak-* compactness in L^∞. Normalizing the approximating eigenfunctions so that $\langle \Phi_n, h_0 \rangle = 1$, and passing to the weak-* limit yields

$$\langle \Phi, h_0 \rangle = 1,$$

hence $\Phi \neq 0$. Moreover,

$$\mathcal{L}^* \Phi = s(\mathcal{L})\Phi.$$

□

Theorem 6.8 (Positive eigenmeasure associated with the L^1-spectral bound)**.** *Let $\Omega \subset \mathbb{R}^d$ be compact with positive Lebesgue measure. Assume that $k \geq 0$ satisfies* (K1) *and that γ satisfies* (Γ1)*. Define*

$$Kf(x) := \int_\Omega k(x,y) f(y)\, dy, \qquad \Gamma f(x) := \gamma(x) f(x),$$

and let $s := s_{L^1}(K - \Gamma)$.

Then there exists a positive finite Radon measure $\mu \in \mathcal{M}_+(\Omega)$ such that $\mu(\Omega) = 1$ and

$$K\mu - \Gamma\mu = s\,\mu$$

in the sense of measures.

Proof. Set $L := K - \Gamma$. Since L generates a positive semigroup, $s \in \sigma(L)$ and $R(\lambda, L) \geq 0$ for every $\lambda > s$. Choose $\lambda_n \downarrow s$ and $g_n \geq 0$, $\|g_n\|_{L^1} \leq 1$, such that $f_n := R(\lambda_n, L)g_n$ satisfies $\|f_n\|_{L^1} \to +\infty$. Define

$$\mu_n(dx) := \frac{f_n(x)}{\|f_n\|_{L^1}}\,dx, \qquad \varepsilon_n := \frac{g_n}{\|f_n\|_{L^1}}.$$

Then $\mu_n \geq 0$, $\mu_n(\Omega) = 1$ and $\|\varepsilon_n\|_{L^1} \to 0$. The resolvent equation gives

$$(\lambda_n + \Gamma)\mu_n = K\mu_n + \varepsilon_n\,dx.$$

By weak-* compactness of probability measures on the compact metric space Ω, up to extraction, $\mu_n \xrightarrow{w^*} \mu \in \mathcal{M}_+(\Omega)$. Testing against $\varphi \equiv 1$ gives $\mu(\Omega) = 1$.

For every $\varphi \in C(\Omega)$, the function

$$K^*\varphi(y) := \int_\Omega k(x, y)\varphi(x)\,dx$$

belongs to $C(\Omega)$ by (K1). Thus, using the continuity of γ, weak-* convergence and $\|\varepsilon_n\|_{L^1} \to 0$, we may pass to the limit in the resolvent equation and obtain

$$\int_\Omega (s + \gamma)\varphi\,d\mu = \int_\Omega K^*\varphi\,d\mu = \int_\Omega \varphi\,d(K\mu).$$

Hence $(s + \Gamma)\mu = K\mu$, which proves the result. □

Proposition 6.9. *Let $\Omega = [0, 1]$, $\gamma(x) = x$, and $k(x, y) = \frac{x}{x+y}$. Then*

$$\lim_{\lambda \downarrow 0} r\big((\lambda + \Gamma)^{-1}K\big) = \lim_{\lambda \downarrow 0} r\big(K(\lambda + \Gamma)^{-1}\big) = \pi.$$

Consequently, $s(L_m) > 0$ if $m < 1 - \frac{1}{\pi}$, whereas $s(L_m) = 0$ if $m \in [1 - \frac{1}{\pi}, 1)$. Hence $m_\star = 1 - \frac{1}{\pi}$.

Proof. For $\lambda > 0$, set $T_\lambda = (\lambda + \Gamma)^{-1}K$. Then

$$(T_\lambda f)(x) = \frac{x}{x + \lambda}\int_0^1 \frac{f(y)}{x + y}\,dy.$$

Let $(Hf)(x) = \int_0^1 \frac{f(y)}{x+y}\,dy$ and $w_\lambda(x) = \frac{x}{x+\lambda}$. Thus $T_\lambda = M_{w_\lambda}H$.

The sharp Hilbert integral inequality [25, Chapter IX, Inequality 9.1.2], together with the dilation invariance of the kernel $(x + y)^{-1}$, gives $\|H\|_{\mathcal{L}(L^2(0,1))} = \pi$. Moreover, $T_\lambda : L^1(0, 1) \to L^\infty(0, 1)$ for every $\lambda > 0$. Since T_λ is compact, its nonzero spectrum is therefore the same whether it is considered on $L^1(0, 1)$ or $L^2(0, 1)$.

On $L^2(0, 1)$, the nonzero spectrum of T_λ coincides with that of $S_\lambda = M_{\sqrt{w_\lambda}}HM_{\sqrt{w_\lambda}}$. Since S_λ is self-adjoint, $r(T_\lambda) = \|S_\lambda\|$. Now $M_{\sqrt{w_\lambda}} \to I$ strongly on $L^2(0, 1)$, so that $S_\lambda \to H$ strongly. Since $\|S_\lambda\| \leq \|H\| = \pi$, we have $\limsup_{\lambda \downarrow 0} \|S_\lambda\| \leq \pi$. Conversely, for every $f \in L^2(0, 1)$ with $\|f\|_2 = 1$,

$$\|Hf\|_2 = \lim_{\lambda \downarrow 0} \|S_\lambda f\|_2 \leq \liminf_{\lambda \downarrow 0} \|S_\lambda\|,$$

and taking the supremum over f yields the reverse inequality. Hence $\lim_{\lambda \downarrow 0} r(T_\lambda) = \pi$. Since for every $\lambda > 0$ we have $r(T_\lambda) = r\left(K(\lambda + \Gamma)^{-1}\right)$ (see [16, Lemma 2.1]), we have the result.

Applying the spectral characterization established above to $L_m = (1 - m)K - \Gamma$ gives $s(L_m) > 0 \iff (1 - m)\pi > 1$. Since $\underline{\gamma} = 0$, the spectral bound is equal to 0 otherwise. Therefore $s(L_m) = 0 \iff m \geq 1 - \frac{1}{\pi}$, which proves the result. □

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